UNIT 12 Number Patterns and Sequences

NC: Algebra 2b

		St	Ac	Ex	Sp
TOPICS (Text and Practice Books)					
12.1	*Simple Number Patterns*	✓	-	-	-
12.2	*Recognising Number Patterns*	✓	✓	-	-
12.3	*Extending Number Patterns*	✓	✓	✓	✓
12.4	*Formulae and Number Patterns*	✓	✓	✓	✓
12.5	*General Laws*	✗	✗	✓	✓
12.6	*Quadratic Formulae*	✗	✓	✓	✓
Activities					
12.1	*Lines*	✓	✓	✓	✓
12.2	*Regular Polygons**	✓	✓	✓	✓
12.3	*Towers*	✗	✓	✓	✓
12.4	*Ulam's Sequence**	✓	✓	✓	✓
12.5	*Bode's Law*	✗	✗	✓	✓
12.6	*Fibonacci Sequence**	✗	✗	✓	✓
OH Slides					
12.1	*Sequences*	✓	✓	-	-
12.2	*Flower Beds*	✓	✓	✓	✓
12.3	*Quadratic Sequences*	✓	✓	✓	✓
12.4	*Linear Formulae*	✗	✓	✓	✓
12.5	*Quadratic Formulae*	✗	✓	✓	✓
Revision Tests					
12.1		✓	-	-	-
12.2		✗	✓	-	-
12.3		✗	✗	✓	✓
Mental Tests					
12.1	(Practice)	✓	✓	-	-
12.2	(Practice)	✓	✓	-	-
12.3		✓	✓	✓	✓
12.2		✓	✓	✓	✓

UNIT 12 *Number Patterns and Sequences* Teaching Notes

Background and Preparatory Work

Work on sequences and number patterns has many attractions, but also many pitfalls.

The human brain is effective precisely because it always tries to 'make sense' of the partial information provided by sensory data by 'filling in' missing details. Thus we act, not on the basis of hard facts, but to a large extent on 'guesses' which extrapolate from partial sensory information. For example, we 'expect' the rustle of letters coming through the letter box purely on the basis of the sound of footsteps coming up the path at a particular time of day. Unfortunately, this tendency depends on *inference* (a fancy name for 'jumping to conclusions'), rather than *deduction*. Our ability to draw plausible inferences from partial information (on the basis of previous experience) – and the way we sharpen this ability is by learning from those occasions when our inferences are incorrect – is an important part of the way we survive in a dangerous world. 'Pattern spotting' is an attempt to exploit this natural human activity for educational purposes.

Unfortunately, there are two aspects of this educational strategy which too often combine to undermine its intended benefits.

(a) The first highlights a serious pitfall, which needs to be understood and carefully avoided if 'pattern-spotting' is to support, rather than to distort, students' mathematical learning.

> *Inference* is based on *guesswork* and so is subjective and illogical, whereas mathematics is about exact calculation, logical deduction, and being absolutely sure.

Thus, though 'pattern-spotting' is a natural human activity which cannot be suppressed, the fact that it is subjective rather than objective means that it is not automatically a mathematical activity. Careful thought is needed if students are to understand the advantages and limitations of 'pattern-spotting', and the fact that guesses are completely unreliable if they are not subsequently justified strictly (by exact calculation).

(b) The difficulty of avoiding this pitfall is accentuated by the fact that, in ordinary human experience, one learns the limitations of inference by suffering the consequences of making painful mistakes. When a child trusts a branch which is too weak to bear its weight, the consequences can be felt for days afterwards! Thus, inferences which are flawed lead to consequences which force us to reconsider how we weigh the sensory information available to us when making guesses.

Educational settings consistently fail to mimic the real world in this respect. Yet any effective exploitation of the human instinct for 'pattern-spotting' must somehow ensure that false inferences have consequences which lead to self-correction – and this must be done before one can even begin to address the tension (between subjective 'pattern-spotting' and objective mathematics) raised in (a) above.

Faced with the first few terms '1, 2, 3, ...' of a number sequence, the human instinct for 'pattern-spotting' immediately suggests a range of possible ways in which the sequence might continue. The extent of this 'space of possibilities' will naturally depend on the experience of the observer. However, mathematics should never be presented as a mere party game in which the student tries to guess what is in the teacher's, or the examiner's, mind. Mathematics is about patterns which occur in the mathematical universe. Thus there are just two possibilities:

(i) On the one hand, one may be working with pure number sequences, in which the rule for generating the sequence is given completely.

Example 1 (u_n) is the sequence for which

$$u_n = n^3 - 6n^2 + 12n - 6.$$

Example 2 (u_n) is the sequence for which

$$u_1 = 1, \ u_2 = 2 \text{ and } u_{n+1} = u_n + u_{n-1}, \text{ for } n \geq 2.$$

In such a setting, there is no need to guess: one only needs to calculate to see how each sequence continues.

(ii) The other possibility is that the sequence 'comes from somewhere' – that is, the *n*th term in the sequence is defined in terms of some configuration which depends on a parameter '*n*'.

Example 3 u_n counts the number of ways of making up exactly *n* pence worth of stamps using only 1p, 2p and 3p stamps.

Example 4 u_n counts the number of different ways of climbing a staircase with *n* steps by taking either one or two steps at a time.

Example 5 u_n counts the number of married couples who can be seated at a long table with *n* chairs down each side and one at each end.

In such a setting, it is natural to calculate the first few terms in the hope of gaining some insight into what exactly is going on, but one must remember clearly that the data one generates is partial and that the final arbiter as to what is going on is the original definition of the *n*th term, u_n.

The first few terms of the sequence in *Example 3* certainly look familiar:

$$u_1 = 1, \ u_2 = 2 \text{ ('1+1' and '2')}, \ u_3 = 3 \text{ ('1+1+1', '1+2', and '3')}.$$

However, one must remember that u_4 is defined as in *Example 3*, not by my personal favourite way of continuing the partial number sequence '1, 2, 3, . . .',

This does not simply mean that one has to 'check for more terms' before one can be sure. The logic is exactly the reverse! Checking further terms may *rule out* certain possibilities, but can never constitute mathematical *proof* that any particular guess is correct.

Thus, provided one calculates carefully, the fourth term in *Example 3* proves that the sequence in *Example 3* is different from that in each of the other examples, but tells you nothing definite about how the sequence in *Example 3* continues. It may *suggest* more than this but it guarantees nothing.

One final remark – it can be helpful to see certain aspects of elementary mathematics as consisting of two complementary stages, with the second stage being consistently harder than the first. Addition needs to be taught with one eye on subtraction; multiplication can be seen as but an introduction to numerical factorisation and division; multiplying out brackets is a precursor to algebraic factorisation. In each case, success at the second stage not only depends on achieving fluency at the first stage, but is considerably more difficult. This is largely because, whilst the first stage is deterministic and correct, the second stage is more elusive and indirect. For example, subtraction depends on scanning a range of possibilities from one's experience of addition to see which one fits the bill.

"$29 - 13 = ?$ Hmm! Nine take away three.
What do I have to add to 3 to get 9?")

In the present context, the *direct* stage may be taken as proceeding from a suitable defined sequence (as in *Examples 1–4* above) to calculating the first few terms. In particular, the direct stage should stress that:

- the most mathematically satisfactory form is the *algebraic closed form* in *Example 1* (which allows one to write down the 100th, 1000th or *n*th term);
- the next most satisfactory form is the *recurrence relation* in *Example 2* (which allows one to generate the sequence very easily step by step, though it does not provide a formula for the *n*th term).

The *indirect* stage builds on this direct experience by trying to 'identify' in algebraic terms sequences which are given in one of the more elusive ways *(Examples 3* and *4*). Thus one has to guess and then prove that such a sequence satisfies either a *closed formula* (highly satisfactory) or, failing that, a *recurrence relation* (not quite so satisfactory).

The important thing to bear in mind when trying to prove that a sequence defined in terms of some configuration with parameter '*n*' (as in *Examples 3–5*) is that the reasoning has to be based on the definition of u_n, and not on the values which u_n takes for small values of n. Given a table of values for a sequence such as that in *Example 3*,

n	1	2	3	4	5	6	7	8	9	...
u_n	1	2	3	4	.	.	.	.	.	...

one must distinguish carefully between the definition of the sequence u_n in terms of n. In contrast, in *Example 5*:

n	1	2	3	4	5	6	7	8	9	...
u_n	2	3	4	.	.	.	.	.	.	...

it may *seem* obvious that the *n*th term is $n+1$ but far too often the 'proofs' offered make no reference to the definition and are merely based on the presumed pattern in the values in the bottom row:

"The numbers go up in 1s."

instead of

"There are 'n chairs down each side (= $2n$) and one at each end (= 2),

$\therefore$ there are $2n + 2 = 2(n+1)$ chairs altogether,

$\therefore$ $n+1$ couples can be seated.".

Teaching Points

Introduction

Most students will be used to finding the next term in a sequence of numbers but will not be familiar with finding the *n*th term. For many coursework tasks, generalising a pattern by finding the *n*th term, is an important part of the project, enabling candidates to reach the higher national curriculum levels.

It is, though, fundamental to any higher mathematical work, that students do understand the difference between

- conjecture (something which you think, but do not actually know, is true);
- generalisation (using the obvious explanation for a pattern in number of shapes in order to predict the *n*th member);
- proof (which shows that a conjecture is true).

So verifying that a generalisation is true for a particular value of n does not constitute a proof! As an example, you might like to discuss the following with students on the *Express/Special* routes:

> Here is a formula for prime numbers:
>
> $$p = n^2 - n + 11$$
>
> It works for $n = 1, 2, 3, 4, 5, 6,$ so it must work for all values of n.

Clearly it does not work for $n = 11$!

Again, for *Express/Special* students in particular, it should be noted that, for example, when asked the next term in the sequence

$$5,\ 10,\ 15,\ ?$$

the obvious answer expected (20) need not be the only one. For example, if the sequence had been generated by

$$u_n = n^3 - 6n^2 + 16n - 6,$$

then

$$u_1 = 5,\ \ u_2 = 10,\ \ u_3 = 15 \text{ but } u_4 = 26!$$

Of course this may appear to be a rather pure mathematical point but we do need to be aware that sometimes what seems obvious is not so obvious at all. To reassure you, in this Unit we are always looking for the most obvious answer but occasionally there may be some ambiguity.

Language / Notation

It is important for pupils to realise that the position in the sequence is denoted by n, and that a general sequence is

$$u_1,\ u_2,\ u_3,\ \ldots,\ u_n,\ \ldots$$

where u_n is called the nth term.

Finally, the rule that gets from one number to the next OS 12.1 and 12.3

(e.g. add 2, take away 5)

is not the same as giving the general formula

(e.g. $u_n = 1 + 2n$, $u_n = 20 - 5n$).

Key Points

- n is the position in the sequence.
- u_n denotes the nth term.
- A linear sequence is given, in general, by OS 12.4

$$u_n = an + b \quad (n = 1, 2, 3, \ldots)$$

- A quadratic sequence is given, in general, by OS 12.5

$$u_n = an^2 + bn + c \quad (\text{n} = 1, 2, 3, \ldots)$$

The importance of having a general formula is that it enables you to calculate any term in the sequence without having to calculate all intermediate terms,

e.g. "Find the 100th term of the sequence: 1, 2, 4, 7, 11, ..."

We could write out all the terms, increasing the differences by 1 each time, but using $u_n = \frac{1}{2}\left(n^2 - n + 2\right)$ gives $u_{100} = 4951$ almost immediately.

Misconceptions

Understanding suffix notation and substituting n = 1, 2, 3, ... into the formula for u_n can lead to confusion. It should be stressed that this notation is introduced to help in generalising (see the example in the *Key Points* above).

It should also be noted that there are many other sequences apart from *linear* (i.e. constant difference) or *quadratic* (i.e. constant second difference). Here are a few examples:

$2,\ 4,\ 8,\ 16,\ 32,\ \ldots \quad \left(u_n = 2^n\right)$

$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ \ldots$ (Fibonacci, $\left(u_n = u_{n-1}, u_{n-2}, ...\right)$ A 12.6

$1,\ 8,\ 27,\ 64,\ 125,\ \ldots \quad \left(u_n = n^3\right)$

$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \ldots \quad \left(u_n = \frac{1}{n}\right)$