# UNIT 7 Mensuration *NC: Shape, Space and Measures 1b, 2a, 2b, 4a-d*

## TOPICS (Text and Practice Books)

| | | St | Ac | Ex | Sp |
|---|---|---|---|---|---|
| 7.1 | *Units and Measuring* | ✓ | - | - | - |
| 7.2 | *Estimating Areas* | ✓ | - | - | - |
| 7.3 | *Making Solids Using Nets* | ✓ | - | - | - |
| 7.4 | *Constructing Nets* | ✓ | ✓ | - | - |
| 7.5 | *Conversion of Units* | ✓ | ✓ | - | - |
| 7.6 | *Squares, Rectangles and Triangles* | ✓ | ✓ | - | - |
| 7.7 | *Area and Circumference of Circles* | ✓ | ✓ | - | - |
| 7.8 | *Volumes of Cubes, Cuboids, Cylinders and Prisms* | ✓ | ✓ | - | - |
| 7.9 | *Plans and Elevations* | ✓ | ✓ | - | - |
| 7.10 | *Using Isometric Paper* | ✓ | ✓ | - | - |
| 7.11 | *Discrete and Continuous Measures* | ✓ | ✓ | ✓ | ✓ |
| 7.12 | *Areas of Parallelograms, Trapeziums, Kites and Rhombuses* | ✓ | ✓ | ✓ | ✓ |
| 7.13 | *Surface Area* | ✓ | ✓ | ✓ | ✓ |
| 7.14 | *Mass, Volume and Density* | ✓ | ✓ | ✓ | ✓ |
| 7.15 | *Volumes, Areas and Lengths* | - | ✓ | ✓ | ✓ |
| 7.16 | *Dimensions* | - | ✓ | ✓ | ✓ |
| 7.17 | *Areas of Triangles* | - | - | ✓ | ✓ |

## Activities (* particularly suitable for coursework tasks)

| | | St | Ac | Ex | Sp |
|---|---|---|---|---|---|
| 7.1 | *Tangram* | ✓ | ✓ | - | - |
| 7.2 | *Closed Doodles* | ✓ | ✓ | ✓ | ✓ |
| 7.3* | *Map Colouring* | ✓ | ✓ | ✓ | ✓ |
| 7.4* | *Euler's Formula* | ✓ | ✓ | ✓ | ✓ |
| 7.5 | *Square-based Oblique Pyramid* | ✓ | ✓ | - | - |
| 7.6 | *Klein Cube* | ✓ | ✓ | ✓ | ✓ |
| 7.7* | *Transforming Polygons* | ✗ | ✓ | ✓ | ✓ |
| 7.8* | *Tubes* | ✗ | ✓ | ✓ | ✓ |
| 7.9* | *Minimum Wrapping* | ✗ | ✓ | ✓ | ✓ |
| 7.10* | *Fence it off* | ✓ | ✓ | ✓ | ✓ |
| 7.11* | *Track Layout* | ✗ | ✓ | ✓ | ✓ |
| 7.12 | *Dipstick Problem* | ✗ | ✗ | ✓ | ✓ |

# TOPICS

| | | St | Ac | Ex | Sp |
|---|---|---|---|---|---|
| **OH Slides** | | | | | |
| 7.1 | *Number Scales* | ✓ | ✓ | - | - |
| 7.2 | *Area* | ✓ | ✓ | - | - |
| 7.3 | *Imperial Units* | ✓ | ✓ | - | - |
| *7.4* | *Conversion Facts* | ✓ | ✓ | - | - |
| *7.5* | *Areas of Rectangles* | ✓ | ✓ | - | - |
| 7.6 | *Areas of Triangles* | ✓ | ✓ | - | - |
| 7.7 | *Areas of Parallelograms* | ✓ | ✓ | ✓ | ✓ |
| 7.8 | *Equal Perimeters* | ✓ | ✓ | ✓ | ✓ |
| 7.9 | *Equal Areas* | ✓ | ✓ | ✓ | ✓ |
| *7.10* | *Volumes* | ✓ | ✓ | ✓ | ✓ |
| *7.11* | *Areas* | ✓ | ✓ | ✓ | ✓ |
| *7.12* | *More Volumes* | ✗ | ✓ | ✓ | ✓ |
| **Mental Tests** | | | | | |
| 7.1 | | ✓ | ✓ | - | - |
| 7.2 | | ✓ | ✓ | - | - |
| **Revision Tests** | | | | | |
| 7.1 | | ✓ | - | - | - |
| 7.2 | | ✓ | ✗ | - | - |
| 7.3 | | ✗ | ✓ | - | - |
| 7.4 | | ✗ | ✓ | - | - |
| 7.5 | | ✗ | ✗ | ✓ | ✓ |

# UNIT 7 *Mensuration* Teaching Notes

## Background and Preparatory Work

Once a child – or a civilisation! – formalises the art of *counting*, it is but a short step to using whole numbers to quantify *measures*. The language and structures we use to refer to large positive integers (say up to low thousands) combine fairly naturally with the idea of a given *unit* to create a flexible and powerful way of quantifying amounts. Thus, for smallish **lengths** the ancient civilisations of the near east (Babylonian/Egyptian, c.1700 BC) used *cubits*; one *cubit* referred to the length of the forearm, or *ulna* (so is related to the later English unit – the *ell*). The ancient Greeks and Romans (400 BC to 600 AD) used *palms* – similar to our *hands,* still used for measuring the height of horses; for longer distances they used the *pes* (or foot), the *passus* (equal to 5 *pedes*), and the *stadium* (roughly a furlong). Many ancient cultures measured **volumes** of grain in *basketfuls.*

The most significant mathematical feature of these early *measures* is that although the units themselves may be inexact (What exactly is a *foot*? What is a *basket?*), the number of units is *absolutely exact* (because we are dealing with whole numbers).

The whole idea of using numbers to quantify amounts has two parts:

- The first part is the mathematical **idea** of choosing a fixed *unit* and then replicating that unit to match a given amount, which can then be assigned a certain number of units, or quantity. This **idea** is *abstract* and *exact.*
- The second part is the **practical implementation** of this scheme, by agreeing
  (a) how to realise the abstract idea of the fixed unit *in practice*; and
  (b) how to replicate the unit *reliably* and *fairly*.
  This **practical implementation** is inevitably approximate.

It is important to establish these two ideas (one exact, and one approximate) in pupils' minds as separate aspects of measurement.

Introducing partial units (halves and other fractions), raises a new source of approximation: it is tempting to think that *one complete basket* involves no approximation, whereas *two thirds of a basket* clearly involves a degree of estimation. This can add to the confusion as to what is exact and what is approximate – especially if one is unclear about the exact nature of fractions.

Units of length, weight, volume and currency developed locally, so only had to be sufficiently accurate for local needs. Trade between regions encouraged the development of common measures, but without the necessary political interest, change was inevitably slow. Moreover, units of measure could never be more accurate than the available technology allowed. The imposition of *standard units of measure* was at the mercy of political and technological developments. The most striking example is the spread of the *metric* *system.* The planning and introduction of the metric system (in France in the 1790s, and thence into other European countries conquered by Napoleon), was the result of a unique combination of events: namely the rise of a powerful Emperor (Napoleon), who happened to be scientifically educated (being a member of the *Académie des Sciences*, and a keen amateur mathematician) at a time when the necessary technological developments were in place for the first time. (For example, the definition of a *metre* as 'the distance between two marks on a specified platinum bar, stored at a fixed temperature' in a vault in Paris would have been unthinkable – for scientific, technological and political reasons – 100 years earlier.)

The kind of practical developments indicated above tended to obscure the simple mathematical idea which underlies the introduction of standard units – namely:

- Once a unit $u$ is chosen as a basic measure for a quantity (e.g. *length*), we can *in principle* measure any other amount $A$ of the same kind (i.e. another *length*) by a number $x$ using the idea of *proportion*.

> the quantity $A$ is measured *exactly* by the number $X$,
>
> whenever
>
> the ratio 'amount $A$ : unit $u$' corresponds to the ratio '$x$ : 1'.

This principle is as fundamental today as it was when first enunciated by the ancient Greeks. However, our trust in improved technology can lead to confusion between the *exactness in principle* of the idea behind measurement and the *inevitable inaccuracy* of all practical measurements. (We tend to use the misleading expression 'accurate' measurement when we really mean that the inevitable error is small!) The key idea behind all measurement is that

once we choose a particular segment $u$ as our unit of length

(be it a *centimetre*, an *inch*, or just an unspecified *unit*),

any other segment $A$ is measured *exactly in principle* by the number $x$ of times that the segment $u$ 'fits in to $A$'.

It is this *idea* that is *exact*; in practice measurement introduces its own inexactness. Thus that the act of measuring is best seen as obtaining precise upper and lower estimates for the *exact* measure: when we say that the diagonal of a square with sides of length 1 cm has length '1.41 cm', we are really saying that the true length lies between 1.405 cm and 1.415 cm.

Approximate practical measurement is quite different from the mathematical fact that a square of side length 1 unit has diagonal of length *exactly* $\sqrt{2}$ units. This result is *exact* because the answer is *calculated* rather than *measured*.

Another important mathematical notion related to units of measurement is the fact that once a unit of *length* is chosen, this gives rise to a natural related unit of *area* (namely the 2-dimensional 'magnitude' of a 1 by 1 square), and to a natural unit of *volume* (namely the 3-dimensional 'magnitude' of a 1 by 1 by 1 cube).

Moreover, once a unit of *time* is chosen, we can combine this with our unit of length to get related units of *speed* and *acceleration*. Similarly, once a unit of *mass* is fixed, we get a related unit of *density*.

Many civilisations developed rules of thumb to find approximate areas and volumes of familiar everyday shapes (such as the area of special shaped fields, and the volume of special grain containers). However, since they usually felt no need to give precise definitions of those shapes, it is often impossible to tell how good their rules were. More detailed procedures – with some proofs – occur in the mathematics of ancient India, China and Japan, but it is hard to know exactly when their methods were developed. Again it was the ancient Greeks (around 300 BC) who probably first gave precise definitions, and set up a structure within which they could realise their insistence on *proving* that their rules were correct. They gave strict proofs for all the basic results we now know (area of rectangles, parallelograms, triangles, trapezia; volumes of prisms; area and perimeter of circles – including the amazing fact that the same number $\pi$ appears in both formulas; volumes of cones and pyramids; volumes and surface areas of spheres).

The later books of Euclid's *Elements* are devoted to giving precise geometrical constructions for all the *Platonic solids* (also called the *regular polyhedra*). Euclid understood the hierarchy of dimension: a polyhedron is bounded by polygons (its faces), a polygon is bounded by line segments (its edges), and a line segment is bounded by its points (its ends). Euclid proves that certain regular polygons (in particular those with 3, 4, 5, 6, 8, 10 sides) can be constructed with ruler and compasses only. He then proves that there are at most five *regular polyhedra* (whose faces are all regular polygons of the same kind, with the same number at each vertex), and shows how to construct each one using ruler and compasses. Archimedes (280?–212 BC) considered polyhedra whose faces are regular, but of more than one type. This work was subsequently developed by Johannes Kepler around 1600 AD.

Euclid proved the fact that the angles of a triangle add up to 'two right angles'; the generalisation for polygons with $n$ sides was given explicitly by subsequent commentators (such as Proclus, 5th century AD). In the early 17th century Descartes gave a similar formula for the (solid) angles of a polyhedron. This result is closely related to *Euler's formula* for complex polyhedra '$(V - E + F = 2)$' (18th century).

Most elementary geometrical properties in 2-dimensions reduce to the study of *triangles.* The natural progression – from (lines to) plane figures to solids – seems perfectly natural, but one soon discovers that geometry in 3-dimensions is both harder and more subtle than geometry in 2-dimensions. Fortunately, many interesting configurations in 3-dimensions can be effectively analysed by 2-dimensional methods. Any three points in a 3-dimensional figure determine a triangle, and hence a 2-dimensional *cross-section.* This is the basis of the important idea that one can often calculate distances and angles in 3-dimensional problems by *choosing a suitable cross-section and then using familiar 2-dimensional methods*. That is why 'solving triangles', and the central ideas of congruence and similarity, hold the key to all elementary geometry.

## Teaching Points

*Introduction*

This is one of the largest units, incorporating

*units, nets, isometric drawings, areas and volumes.*

It has its roots firmly based in real objects and we would want to encourage you at all times to make the connection to the real world. We have provided activities based on making 3-D objects, but for the *Standard/Academic* routes you might well want to use other more basic constructions. A7.5, A7.6

The activities in this Unit also provide many suitable opportunities for coursework, again based on contexts in the real world. A7.3-4, A7.7-11

As the Unit is a long one, particularly for the *Standard/Academic* routes, you will note that we have provided two revision tests in order to adequately cover all the material. You might find it convenient to use each test when the relevant material has been covered, rather than leave them both until the end. R7.1 and 7.2, R7.3 and 7.4

*Language / Notation*

This Unit uses extensive mathematical language, and you will need to check that this has been covered adequately. For example, you need to be familiar with

*units* : km, cm, mm; mile, yard, feet and inches
litre, gallon, pint;
kg, g, tonne; lb, oz, stone.

*shapes* : square, rectangle, triangle, cube,
parallelogram, trapezium, kite, rhombus,
cube, cuboid, cylinder, prism, pyramid, cone, sphere.

*Key Points*

- Pupils should be familiar with reading scales and converting units before embarking on the main topics in this Unit. OS7.1
- Isometric drawings and plans and elevations are two alternative ways of representing 3-D objects in 2-D.

*Misconceptions*

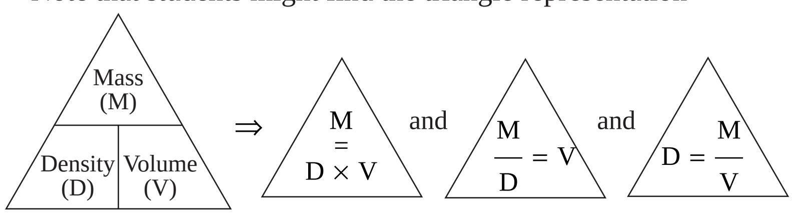
- pupils should realise that squares are special cases of rectangle, rectangles are special parallelograms, etc.

*Key Concepts*

| | St | A | E | Sp |
|---|---|---|---|---|
| 1 $m^3$ = 1000 litres | ✓ | ✓ | - | - |
| 1 tonne = 1000 kg | ✓ | ✓ | - | - |
| 1 gallon = 8 pints | ✓ | ✓ | - | - |
| 1 kg is about 2.2 lbs | ✓ | ✓ | - | - |
| 1 gallon is about 4.5 litres | ✓ | ✓ | - | - |
| 1 litre is about 1.75 pints | ✓ | ✓ | - | - |
| 5 miles is about 8 km | ✓ | ✓ | - | - |
| 1 inch is about 2.5 cm | ✓ | ✓ | - | - |

| | | | | |
|---|---|---|---|---|
| *Square* : Area $= x^2$ | ✓ | ✓ | ✓ | ✓ |
| *Rectangle* : Area $= lw$ | ✓ | ✓ | ✓ | ✓ |
| *Triangle* : Area $= \frac{1}{2}bh$ | ✓ | ✓ | ✓ | ✓ |
| *Circle* : Area $= \pi r^2$<br>Circumference $= 2\pi r$ | ✓ | ✓ | ✓ | ✓ |
| *Parallelogram* : Area $= bh$ | ✓ | ✓ | ✓ | ✓ |
| *Trapezium* : Area $= \frac{1}{2}(a+b)h$ | ✓ | ✓ | ✓ | ✓ |
| *Kite* : Area $= \frac{1}{2}ab$ | ✓ | ✓ | ✓ | ✓ |
| *Cube* : Volume $= a^3$<br>Surface area $= 6a^2$ | ✓ | ✓ | ✓ | ✓ |
| *Cuboid* : Volume $= abc$<br>Surface area $= 2(ab+bc+ca)$ | ✓ | ✓ | ✓ | ✓ |
| *Cylinder* : Volume $= \pi r^2 h$<br>Surface area $= 2\pi r^2 + 2\pi rh$ | ✓ | ✓ | ✓ | ✓ |
| *Prism* : Volume $= Al$ | ✓ | ✓ | ✓ | ✓ |
| *Pyramid* : Volume $= \frac{1}{3}Ah$ | ✗ | ✓ | ✓ | ✓ |
| *Cone* : Volume $= \frac{1}{3}\pi r^2 h$ | ✗ | ✓ | ✓ | ✓ |
| *Sphere* : Volume $= \frac{4}{3}\pi r^3$ | ✗ | ✓ | ✓ | ✓ |
| *Area of triangle* : $A = \frac{1}{2}ab\sin\theta$ | ✗ | ✓ | ✓ | ✓ |
| $A = \sqrt{s(s-a)(s-b)(s-c)}$ | ✗ | ✗ | ✓ | ✓ |
| Density $= \frac{\text{Mass}}{\text{Volume}}$ | ✓ | ✓ | ✓ | ✓ |

Note that students might find the triangle representation




useful in remembering the final result above.