1 Logic

This unit introduces ideas of *logic*, a topic which is the foundation of all mathematics. We will be looking at logic puzzles and introducing some work on sets.

1.1 Logic Puzzles

Here we introduce logic puzzles to help you think mathematically.

Example

Rana, Toni and Millie are sisters. You need to deduce which sister is 9 years old, which one is 12 and which one is 14. You have two clues:

Clue 1 : *Toni's age is not in the 4-times table.*

Clue 2 : *Millie's age can be divided exactly by the number of days in a week.*

Solution

You can present this information in a logic table, shown opposite.

A *cross* in any box means that the statement is *not true*.

A *tick* in any box means that the statement *is true*.

	9 *yrs*	12 *yrs*	14 *yrs*
Rana			
Toni			
Millie			

Clue 1 : *Toni's age is not in the 4-times table.*

This tells you that Toni's age is not 12.

Put a cross in Toni's row and column 12.

	9 *yrs*	12 *yrs*	14 *yrs*
Rana			
Toni		×	
Millie			

Clue 2 : *Millie's age can be divided exactly by the number of days in a week.*

This tells you that Millie's age is 14.

Put 2 crosses and a tick in Millie's row.

	9 *yrs*	12 *yrs*	14 *yrs*
Rana			
Toni		×	
Millie	×	×	✓

Looking at column '12 yrs', you can see that Rana must be 12.

Fill in the ticks and crosses in Rana's row.

	9 *yrs*	12 *yrs*	14 *yrs*
Rana	✗	✓	✗
Toni		✗	
Millie	✗	✗	✓

Looking at column '9 yrs', you can see that Toni must be 9.

Toni's row can now be completed.

	9 *yrs*	12 *yrs*	14 *yrs*
Rana	✗	✓	✗
Toni	✓	✗	✗
Millie	✗	✗	✓

Answer : Toni is 9 years old.

Rana is 12 years old.

Millie is 14 years old.

Exercises

1. Jane, Bill and Kelly each have one pet. They all own different types of pet.

Clue 1: Kelly's pet does not have a beak.

Clue 2: Bill's pet lives in a bowl.

Use this logic table to find out which pet each person owns.

	Goldfish	*Dog*	*Budgie*
Jane			
Bill			
Kelly			

2. Karen, John and Jenny each play one sport: badminton, tennis or football. Use these clues to decide who plays which sport.

Clue 1: John hits a ball with a racket.

Clue 2: Karen kicks a ball.

	Badminton	*Tennis*	*Football*
Karen			
John			
Jenny			

3. Three children are asked to name their favourite subject out of Maths, PE and Art. They each give a different answer. Decide which child names which subject.

Clue 1: Daniel likes working with numbers.

Clue 2: Sarah does not like to draw or paint.

	Maths	*PE*	*Art*
Daniel			
Sarah			
Jane			

4. The three children in a family are aged 8, 12 and 16. Use these clues to find the age of each child.

Clue 1: Alan is older than Charlie.

Clue 2: John is younger than Charlie.

	8 *yrs*	12 *yrs*	16 *yrs*
John			
Alan			
Charlie			

5. A waiter brings these meals to the table in a restaurant.

Chips, steak and salad

Baked potato, cheese and beans

Chips, mushroom pizza and salad

Use the clues to decide who eats which meal.

- *Chris does not eat salad.*
- *Adam is a vegetarian.*

6. Amanda, Jo, Alex and Zarah each have different coloured cars. One car is red, one blue, one white and the other is black.

Decide which person has which coloured car.

- *Amanda's car is not red or white.*
- *Jo's car is not blue or white.*
- *Alex's car is not black or blue.*
- *Zarah's car is red.*

	Red	*Blue*	*White*	*Black*
Amanda				
Jo				
Alex				
Zarah				

7. Bill, John, Fred and Jim are married to one of Mrs Brown, Mrs Green, Mrs Black and Mrs White.

Use these clues and the table to decide who is married to who.

	Bill	*John*	*Fred*	*Jim*
Mrs Brown				
Mrs Green				
Mrs Black				
Mrs White				

Clues

- *Mrs Brown's husband's first name does not begin with J.*
- *Mrs Black's husband has a first name which does have the same letter twice.*
- *The first name of Mrs White's husband has 3 letters*

8. In a race the four fastest runners were Alice, Leah, Nadida and Anna.

Decide who finished in 1st, 2nd, 3rd and 4th places.

- *Alice finished before Anna.*
- *Leah finished before Nadida.*
- *Nadida finished before Alice.*

9. There are 4 children in a family. They are 6, 8, 11 and 14 years old. Use these clues and the table to find out the age of each child.

Clues

- *Dipak is 3 years older than Ali.*
- *Mohammed is older than Dipak.*

	6 years	*8 years*	*11 years*	*14 years*
Ali				
Mohammed				
Dipak				
Nesima				

10. Here is a completed logic table.

	Football	*Tennis*	*Hockey*	*Rugby*
Ben	✓	✗	✗	✗
Tom	✗	✗	✗	✓
Helen	✗	✗	✓	✗
Abbie	✗	✓	✗	✗

(a) Write a set of clues that will give this answer.

(b) Try your clues out on a friend.

1.2 Two Way Tables

Here we extend the ideas of the first section and present data in two way tables, from which we can either complete the tables or deduce information.

Example

Emma collected information about the cats and dogs that children in her class have. She filled in the table below, but missed out one number.

	Has a dog	*Does not have a dog*
Has a cat	8	4
Does not have a cat	12	

(a) Explain how to find the missing number if there are 30 children in Emma's class.

(b) How many children own at least one of these pets?

(c) Do more children own cats rather than dogs?

(d) Could it be true that some of the children do not have any pets?

Solution

(a) As there are 30 children in the class, each one has one entry in the complete table. As there are already

$$8 + 4 + 12 = 24$$

entries, the missing number is

$$30 - 24 = 6$$

	Has a dog	*Does not have a dog*
Has a cat	8	4
Does not have a cat	12	?

(b) All the children, except those in the bottom right hand square, own at least one cat or dog.

	Has a dog	*Does not have a dog*
Has a cat	8	4
Does not have a cat	12	6

Hence,

number of children owning at least one cat or dog is

$$30 - 6 = 24$$

(c) The total number of children owning a dog is given in the first column,

i.e. $8 + 12 = 20$

The total number of children owning a cat is given in the first row,

i.e. $8 + 4 = 12$

	Has a dog	*Does not have a dog*
Has a cat	8	4
Does not have a cat	12	6

So the answer to the question is NO, since there are more dog owners than cat owners.

(d) There are 6 children that do not own either a cat or a dog, but they might own a hamster or rabbit, etc., so we cannot deduce that some children have no pets.

Exercises

1. People leaving a football match were asked if they supported Manchester United or Newcastle. They were also asked if they were happy. The table below gives the results.

	Manchester United	*Newcastle*
Happy	40	8
Not happy	2	20

(a) How many Manchester United supporters were happy?

(b) How many Manchester United supporters were asked the questions?

(c) How many Newcastle supporters were not happy?

(d) How many people were asked the questions?

(e) Which team do you think won the football match? What are your reasons for your answer?

2. The children in a class conducted a survey to find out how many children had videos at home and how many had computers at home. Their results are given in the table.

	Video	*No Video*
Computer	8	2
No Computer	20	3

(a) How many children did *not* have a video at home?

(b) How many children had a computer at home?

(c) How many children did *not* have a computer or a video at home?

(d) How many children were in the class?

3. The children in a school are to have extra swimming lessons if they cannot swim. The table gives information about the children in Years 7, 8 and 9.

	Can swim	*Cannot swim*
Year 7	120	60
Year 8	168	11
Year 9	172	3

(a) How many children need swimming lessons?

(b) How many children are there in Year 8?

(c) How many of the Year 7 children *cannot* swim?

(d) How many children in Years 7 and 8 *can* swim?

(e) How many children are there altogether in Years 7, 8 and 9?

4. 40 children are members of a cycling club. Details of their bikes are given below. Each child has one bike.

	Mountain Bike	*Racing Bike*	*BMX Bike*
15-speed	2	0	0
12-speed	8		0
10-speed	1	8	0
1-speed	0	0	15

(a) How many children have 12-speed racing bikes?

(b) How many children have mountain bikes?

(c) Which type of bike is most popular?

5. The headteacher of a school with 484 pupils collected information about how many of the pupils wear glasses.

	Always wear glasses	*Sometimes wear glasses*	*Never wear glasses*
Boys	40		161
Girls	36	55	144

(a) Explain how to find the number of boys who *sometimes* wear glasses.

(b) How many of the pupils wear glasses some of the time?

(c) How many of the pupils *never* wear glasses?

(d) Are there more boys or girls in the school?

6. During one month, exactly half of the 180 babies born in a hospital were boys, and 40 of the babies weighed 4 kg or more. There were 26 baby boys who weighed 4 kg or more.

	Less than 4 kg	*4 kg or more*
Boys		
Girls		

(a) Copy and complete the table above.

(b) How many baby girls weighed less than 4 kg when they were born?

7. In a school survey pupils chose the TV programme they liked best from a list. Some of the results are given in the table.

	Blue Peter	*Grange Hill*	*Newsround*
Year 7	8		1
Year 8	12	5	

The same number of pupils took part from Year 7 and Year 8. Six pupils chose Newsround. Copy and complete the table and state which programme was the most popular.

8. 18 people who took part in a survey had blue eyes and 22 people had other coloured eyes. In the same survey, 16 people had blond hair and 24 did not have blond hair.

(a) How many people took part in the survey?

(b) Explain why it is impossible to complete the table below.

	Blue eyes	*Not blue eyes*
Blond hair		
Not blond hair		

(c) Complete the table if $\frac{3}{4}$ of the people with blond hair had blue eyes.

(d) How many people did *not* have blond hair and did *not* have blue eyes?

9. In a car showroom there are 8 blue cars, one of which is a hatchback.
If 6 of the 20 cars in the showroom are hatchbacks, find how many cars are not hatchbacks and are not blue.

10. In a class of 32 pupils, there were 8 girls who played hockey and 5 boys who did not. Find how many boys played hockey if there were 15 girls in the class.

1.3 Sets and Venn Diagrams

We use the idea of *sets* to classify numbers and objects and we use *Venn diagrams* to illustrate these sets.

Example

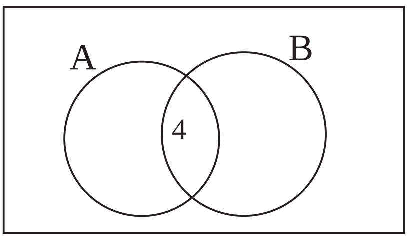
The sets A and B consist of numbers taken from the numbers 0, 1, 2, 3, . . . , 9 so that

Set A = { 4, 7, 9 }

Set B = { 1, 2, 3, 4, 5 }

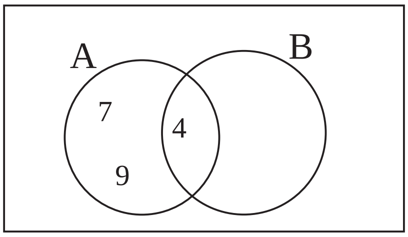
Illustrate these sets in a Venn diagram.

Solution

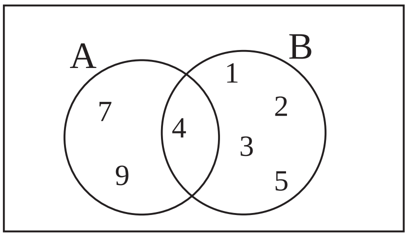


The framework for a Venn diagram is shown opposite, with the sets A and B indicated by the circles.

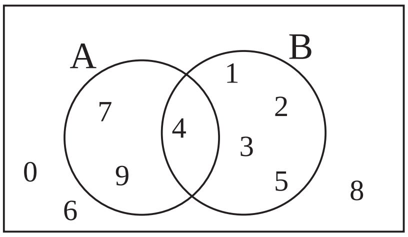
Since 4 is in both sets, it must be placed in the *intersection* of the two sets.



To complete set A, you put 7 and 9 in the part that does not intersect with B.



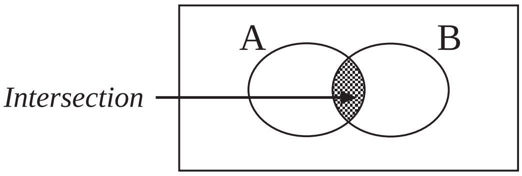
Similarly for B, you put 1, 2, 3 and 5 in the part that does not intersect with A.



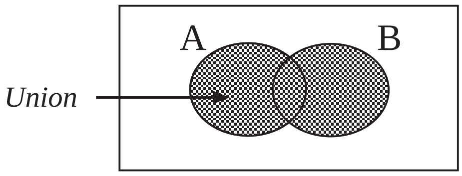
Finally, since the numbers 0, 6 and 8 have not been used in A or B, they are placed outside both A and B.

Note

The *intersection* of two sets consists of any numbers (or objects) that are in both A and B.



The *union* of two sets consists of any numbers (or objects) that are in A or in B or in both.



In the example above,

the intersection of A and B = { 4 }

the union of A and B = { 1, 2, 3, 4, 5, 7, 9 }

Note that, although the number 4 occurs in both A and B, it is *not* repeated when writing down the numbers in the union.
The complement of a set consists of any numbers (or objects) that are not in that set. In the example above,

the complement of A = { 0, 1, 2, 3, 5, 6, 8 }

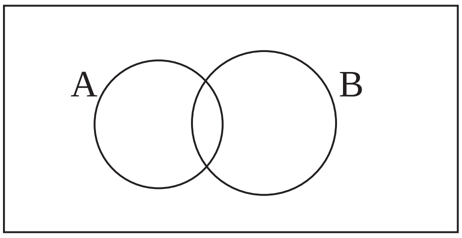
the complement of B = { 0, 6, 7, 8, 9 }

Exercises

1. Set A = { 1, 4, 5, 7, 8 }

   Set B = { 2, 6, 8, 10 }

   (a) Copy and complete the Venn diagram. Include all the whole numbers from 1 to 10.



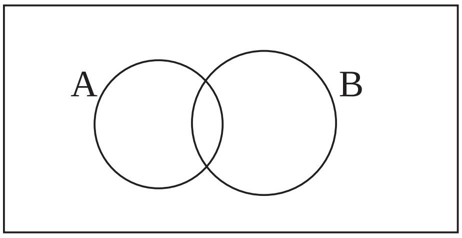
   (b) What is the intersection of A and B?

2. The whole numbers 1 to 10 are organised into 2 sets, set A and set B.

   Set A contains all the odd numbers.

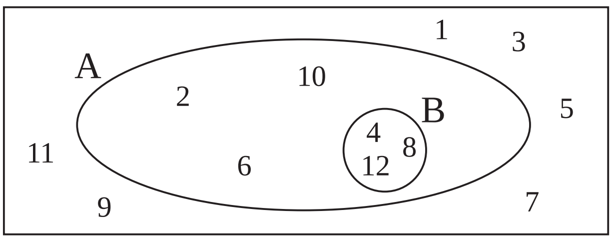
   Set B contains all the numbers greater than 4.

   (a) Copy and complete this diagram.



   (b) What is the union of A and B?

3. The whole numbers 1 to 12 are included in the Venn diagram.



   (a) List set A.

   (b) List set B.

   (c) Describe both sets in words.

   (d) What is the complement of A?

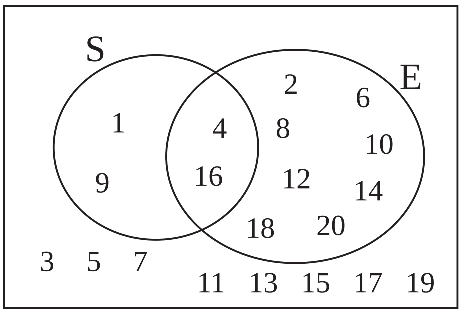
4. (a) Draw a Venn diagram to illustrate the sets P and Q. Include all the whole numbers from 1 to 15 in your diagram.

$P = \{3, 5, 7, 9\}$

$Q = \{1, 3, 5, 7, 9, 11, 13, 15\}$

(b) What is the intersection of P and Q?

5. The whole numbers 1 to 20 are organised into sets as shown in the Venn diagram below.



(a) List set E.

(b) List set S.

(c) Describe each set in words.

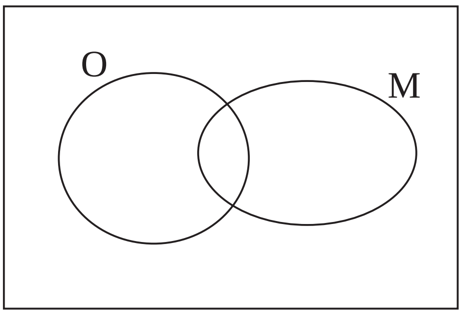
(d) What is the union of E and S?

6. The whole numbers 1 to 20 are organised into two sets,

O : Odd numbers

M: Multiples of 5

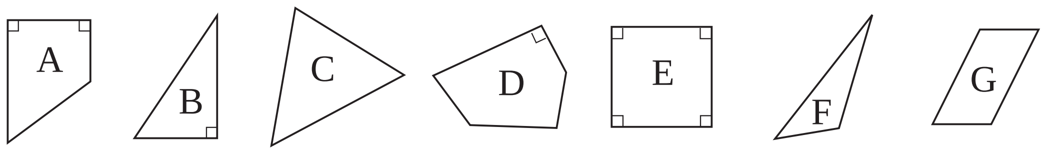
Copy and complete the Venn diagram, placing each number in the correct place.

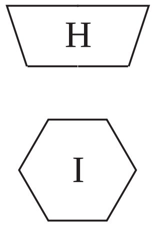


7. The shapes shown below are to be sorted into 2 sets, R and Q.

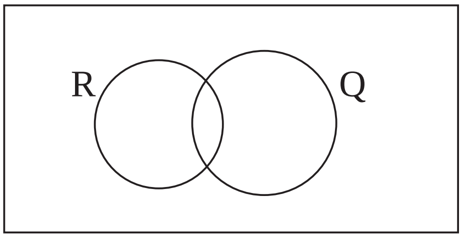
   R contains shapes with a right angle.

   Q contains shapes with four sides.





   (a) Sort the shapes using the Venn diagram below.



   (b) Which shapes are in both sets?

   (c) Which shapes are in R but not in Q?

   (d) Which shapes are not in R or Q?

8. Set P contains the letters needed to spell 'JENNY'.

   Set Q contains the letters needed to spell 'JEN'.

   Set R contains the letters needed to spell 'TED'.

   (a) Draw a Venn diagram for the two sets, P and R.

   (b) Draw a Venn diagram for the two sets, P and Q.

   (c) What is the union of P and R?

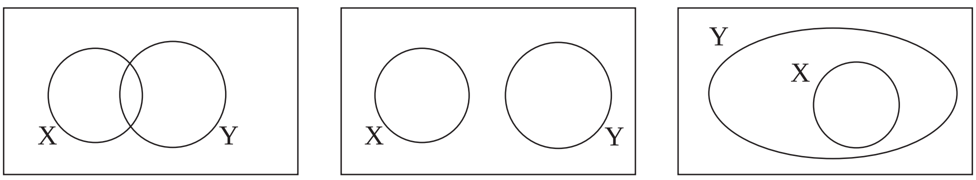
   (d) What is the intersection of P and R?

9. Set S contains silver coins in circulation in the UK.

   Set R contains circular coins in circulation in the UK.

   Draw a Venn diagram to illustrate these two sets. You should include *all* UK coins in the Venn diagram.

10. Which of these Venn diagrams would be best for the sets described below?



A B C

(a) X is the set of all squares.
Y is the set of all rectangles.

(b) X is the set of all triangles.
Y is the set of all squares.

(c) X is the set of all quadrilaterals (4-sided shapes).
Y is the set of all triangles.

(d) X is the set of all shapes containing a right angle.
Y is the set of all triangles.

1.4 Set Notation

We use ξ to denote the *universal* set, that is, the set from which we are picking the members of A, B, ...

$A \cap B$, the intersection of A and B, is the set of members in set A *and* in set B.

$A \cup B$, the union of A and B, is the set of members in set A or in set B or in both.

A', the complement of A, is the set of members in ξ but not in A.

$A \subset B$ means that A is a subset of B, i.e. every element in A is also in B.

$\varnothing$ is the empty set, i.e. the set with no numbers (or objects).

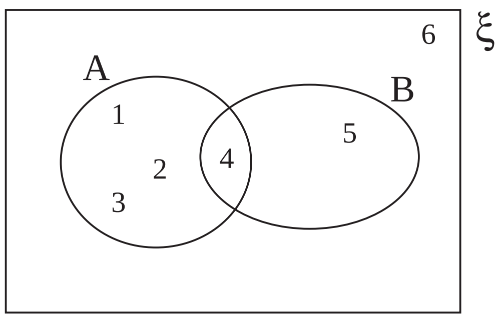
Example 1

If $\xi = \{1, 2, 3, 4, 5, 6\}$ and $A = \{1, 2, 3, 4\}$, $B = \{4, 5\}$

find (a) $A \cap B$, (b) $A \cup B$ (c) A' (d) B'

Is $B \subset A$?

Solution



First put the numbers in a Venn diagram.

(a) $A \cap B = \{4\}$

(b) $A \cup B = \{1, 2, 3, 4, 5\}$

(c) $A' = \{5, 6\}$

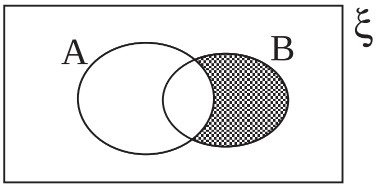
(d) $B' = \{1, 2, 3, 6\}$

No, B is not a subset of A since the number 5 is in B but not in A.

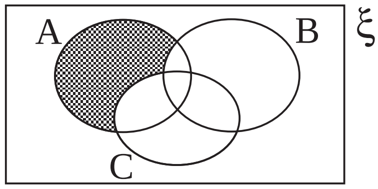
Example 2

Use set notation to describe the shaded regions of these diagrams.

(a)



(b)



Solution

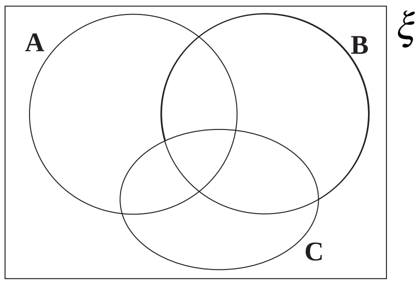
(a) This is the intersection of B with A', i.e. $B \cap A'$.

(b) This is the intersection of A with the complement of the union of B and C, i.e. $A \cap (B \cup C)'$.

Example 3

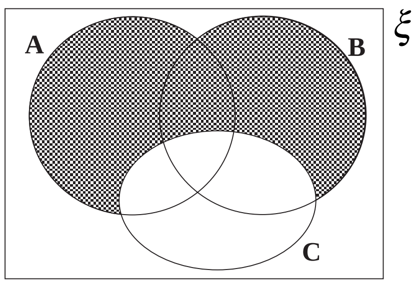
On this diagram, shade the region that represents

$$(A \cup B) \cap C'$$



Solution

You want the union of A and B which is not in C.



Exercises

1. If $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

   $A = \{2, 4, 6, 8\}$

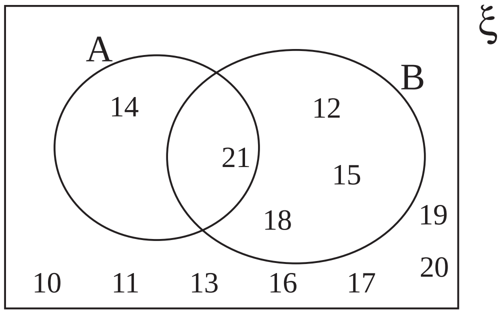
   and $B = \{3, 6, 9\}$

   find:

   (a) $A \cap B$ (b) $A \cup B$ (c) A'

   (d) B' (e) $A' \cap B'$ (f) $A' \cup B'$

2. The Venn diagram illustrates sets A, B and ξ.



Find:

(a) $A \cap B$ (b) $(A \cap B)'$ (c) $A \cup B$

(d) A' (e) B' (f) $A' \cap B'$

(g) $A' \cap B$

3. If $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$A = \{1, 3, 6, 10\}$

$B = \{1, 5, 10\}$

and $C = \{3, 6, 9, 12\}$,

find:

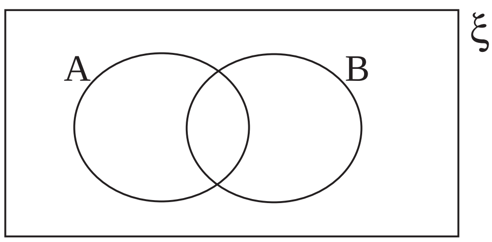
(a) $A \cap B$ (b) $A \cap C$ (c) $B \cap C$

(d) $A \cup B$ (e) $A \cup C$ (f) C'

(g) $A \cap C'$ (h) B' (i) $B' \cup C'$

(j) $A \cap B \cap C$ (k) $A \cup B \cup C$

4. Make a separate copy of this diagram for each part of the question.



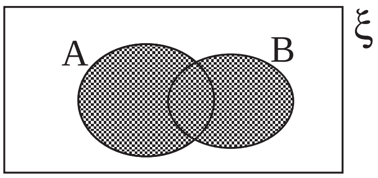
Shade the region on the diagram that represents:

(a) $A \cap B$ (b) A' (c) $A \cup B'$

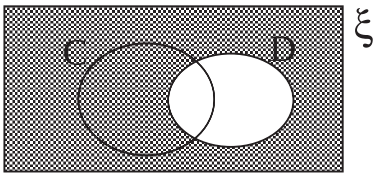
(d) $A' \cap B'$ (e) $A \cap B'$ (f) $(A \cup B)'$

5. Use set notation to describe the region shaded in each of these diagrams.

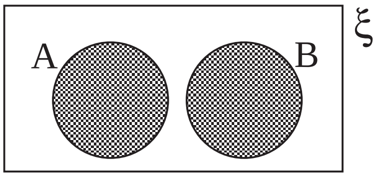
(a)



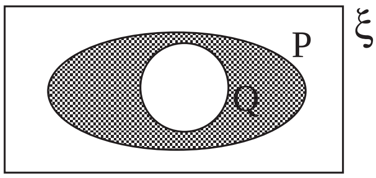
(b)



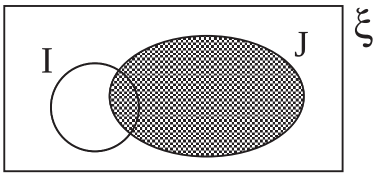
(c)



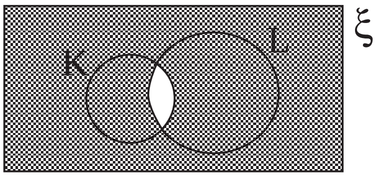
(d)



(e)

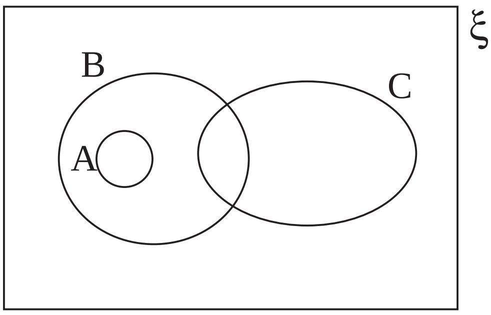


(f)



6. The diagram illustrates 3 sets, A, B and C.

Say whether each of these statements is *true* or *false*.



(a) $A \subset B$

(b) $B \subset C$

(c) $A \cap B = A$

(d) $A \cap B = C$

(e) $A \cap C = \varnothing$

(f) $B \cap C = \varnothing$

(g) $B \cup A = B$

7. If $\xi = \{a, b, c, d, e, f, g, h\}$

$A = \{a, c, e\}$

$B = \{b, d, g, h\}$

and $C = \{a, c, e, f\}$,

say whether each of these statements is *true* or *false*. Write correct statements to replace those that are false.

(a) $B \cap C = \varnothing$

(b) $C \subset A$

(c) $B \cup C = \xi$

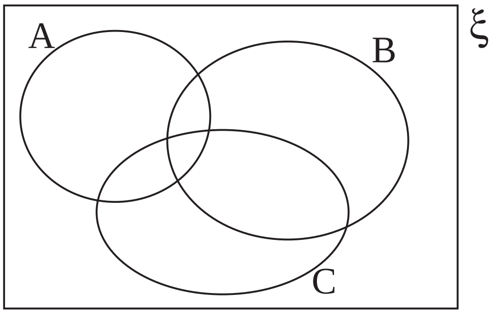
(d) $A \cap C = \{a, c, e, f\}$

(e) $(A \cap C)' = \{b, d, f, g, h\}$

(f) $B \subset \xi$

(g) $A \cap B' = C$

8. For each part of the question, use a copy of the diagram.



Shade the region of the diagram that represents:

(a) $A \cap B \cap C$

(b) $(A \cup B) \cap C$

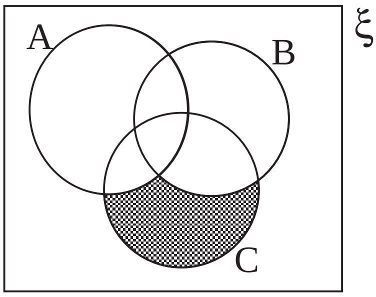
(c) $(A \cap B) \cup C$

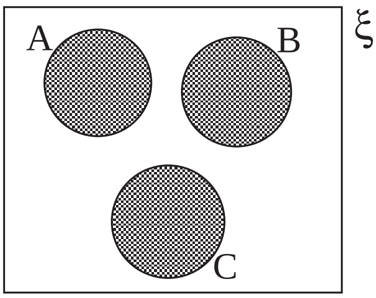
(d) $A' \cap (B \cup C)$

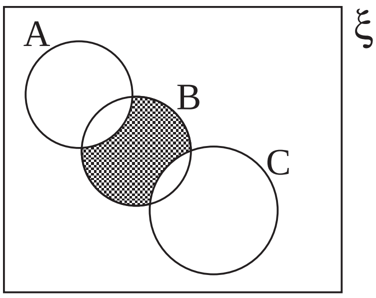
(e) $A' \cap B \cap C$

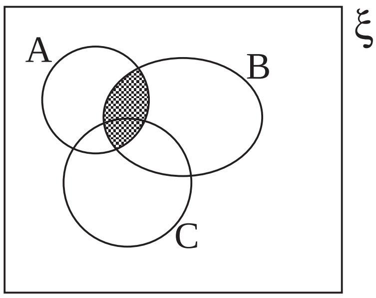
(f) $A' \cap B' \cap C'$

9. Use set notation to describe the regions shaded in each of these diagrams.

(a)


(b)


(c)


(d)


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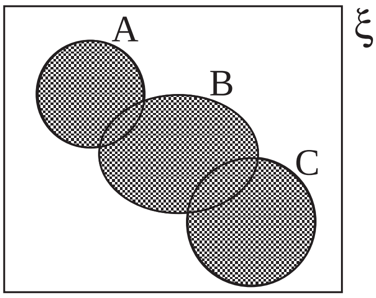
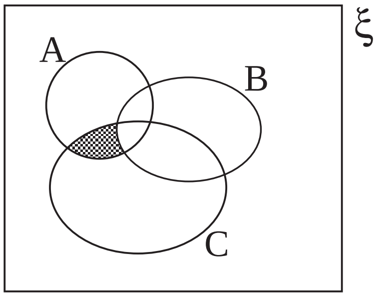


(f)


10. If $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$A = \{1, 3, 5, 7\}$

$B = \{2, 4, 6, 8\}$

$C = \{1, 3, 5, 7, 9\}$,

draw a Venn diagram to represent these sets.

Then find

(a) $A \cap B$ (b) C'

(c) $A \cap C$ (d) $B \cap C$

(e) $A \cup B$ (f) $(A \cup B)'$

(g) $(A \cup B) \cap C$ (h) $(A \cup B) \cap C'$

1.5 Logic Problems and Venn Diagrams

Venn diagrams can be very helpful in solving logic problems.

Example

In a class there are

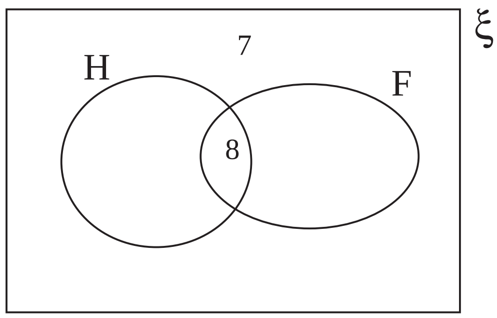
- 8 students who play football and hockey
- 7 students who do not play football or hockey
- 13 students who play hockey
- 19 students who play football.

How many students are there in the class?

Solution

You can use a Venn diagram to show the information.

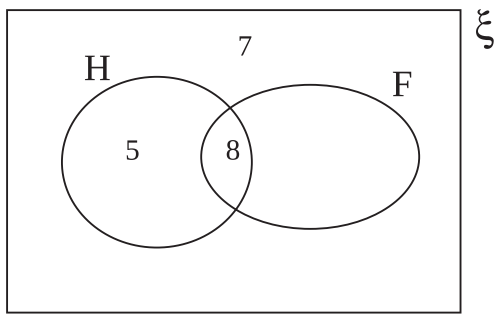
The first two sets of students can be put directly on to the diagram.



If there are 13 students who play hockey, and we already know that 8 play hockey and football, then there must be

$$13 - 8 = 5$$

who play just hockey.



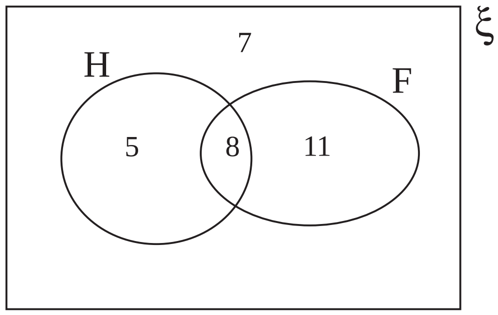
Similarly for football,

$$19 - 8 = 11$$

play just football.

So the total number of students in the class is

$$7 + 5 + 8 + 11 = 31$$

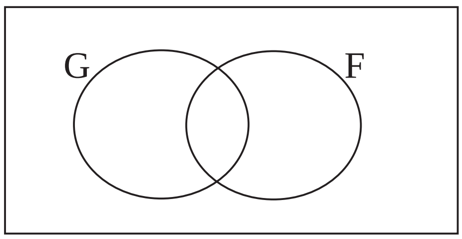


Exercises

1. In a family of six, everybody plays football or hockey. 4 members of the family play both sports and 1 member of the family plays only hockey. How many play only football?

2. John's mum buys 5 portions of chips. All the portions have salt or vinegar on them. Some have salt and vinegar. There are 2 portions with salt and vinegar and one portion with only vinegar. How many portions have only salt on them?

3. This diagram represents a class of children. G is the set of girls and F is the set of children who like football. Make 4 copies of this diagram.



On separate diagrams, shade the part that represents:

(a) girls who like football,
(b) girls who dislike football,
(c) boys who like football,
(d) boys who do not like football.

4. In a class of 32 pupils, 20 say that they like pancakes and 14 say that they like maple syrup. There are 6 pupils who do not like either. How many of them like both pancakes and maple syrup?

5. On a garage forecourt there are 6 new cars, 12 red cars and no others.

(a) What is the maximum possible number of cars on the forecourt?

(b) What is the smallest possible number of cars on the forecourt?

(c) If 2 of the new cars are red, how many cars are on the forecourt?

6. There are 20 people in a room. Of these, 15 are holding newspapers and 8 are wearing glasses. Everyone wears glasses or holds a newspaper. How many people are wearing glasses *and* holding a newspaper?

7. A pencil case contains 20 pens that are red or blue. Of these, 8 are blue and 6 do not work. How many of the blue pens do not work if there are 8 red pens that do work?

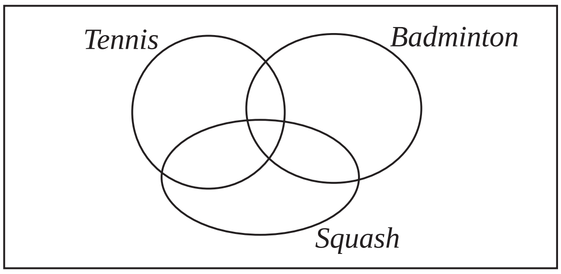
8. In a school canteen there are 45 children. There are 16 who have finished eating. The others are eating either fish or chips, or both fish and chips. There are 26 eating chips and 17 eating fish.

   (a) How many are eating fish and chips?

   (b) How many are eating fish without chips?

   (c) How many are eating only chips?

9. Youth club members can choose to play tennis, badminton or squash. The diagram below represents the possible combinations.



   Make 3 copies of the diagram.

   On separate diagrams shade the parts that represent:

   (a) those who play all three sports,

   (b) those who play tennis and badminton, but not squash,

   (c) those who play only tennis.

10. All the members of a group of 30 teenagers belong to at least one club. There are 3 clubs, chess, drama and art.

    6 of the teenagers belong to only the art club.

    5 of the teenagers belong to all 3 clubs.

    2 of the teenagers belong to the chess and art clubs but not to the drama club.

    15 of the teenagers belong to the art club.

    2 of the teenagers belong only to the chess club.

    3 of the teenagers belong only to the drama club.

(a) How many of the group belong to the chess club and the drama club, but not the art club?

(b) How many of the group belong to each club?

11. In a class of 32 pupils:

5 pupils live in New Town, travel to school by bus and eat school dinners,

3 pupils live in New Town, travel to school by bus but do not eat school dinners.

9 pupils do not live in New Town, do not travel to school by bus and do not eat school dinners.

11 pupils live in New Town and have school dinners.

16 pupils live in New Town.

9 pupils travel by bus and eat school dinners.

13 pupils travel by bus.

How many pupils eat school dinners?