

STRAND: DISCRETE MATHS

Unit 3 *Linear Programming*

TEXT

Contents

Section

- 3.1 Formation of Linear Programming Problems
- 3.2 Graphical Solution
- 3.3 Simplex Method

3 Linear Programming

3.1 Formation of Linear Programming Problems

You are now in a position to use your knowledge of inequalities from Unit 4 of the Algebra strand to illustrate **linear programming** with the following case study.

Suppose a manufacturer of printed circuits has a stock of

200 resistors, 120 transistors and 150 capacitors

and is required to produce two types of circuits.

Type A requires 20 resistors, 10 transistors and 10 capacitors.

Type B requires 10 resistors, 20 transistors and 30 capacitors.

If the profit on type A circuits is £5 and that on type B circuits is £12, how many of each circuit should be produced in order to maximise the profit?

You will not actually solve this problem yet, but show how it can be formulated as a linear programming problem. There are three vital stages in the formulation, namely

- (a) What are the unknowns?
- (b) What are the constraints?
- (c) What is the profit/cost to be maximised/minimised?

For this problem,

- (a) **What are the unknowns?**

Clearly the number of type A and type B circuits produced; so we define

x = number of type A circuits produced

y = number of type B circuits produced

- (b) **What are the constraints?**

There are constraints associated with the total number of resistors, transistors and capacitors available.

Resistors Since each type A requires 20 resistors, and each type B requires 10 resistors, then

$$20x + 10y \leq 200$$

as there is a total of 200 resistors available.

Transistors Similarly

$$10x + 20y \leq 120$$

Capacitors Similarly

$$10x + 30y \leq 150$$

3.1

Finally you must state the obvious (but nevertheless important) inequalities

$$x \geq 0, y \geq 0$$

(c) **What is the profit?**

Since each type A gives £5 profit and each type B gives £12 profit, the total profit is £ P , where

$$P = 5x + 12y$$

You can now summarise the problem as:

$$\text{maximise } P = 5x + 12y$$

$$\text{subject to } 20x + 10y \leq 200$$

$$10x + 20y \leq 120$$

$$10x + 30y \leq 150$$

$$x \geq 0$$

$$y \geq 0$$

This is called a **linear** programming problem since both the objective function P and the constraints are all linear in x and y .

In this particular example you should be aware that x and y can only be integers since it is not sensible to consider fractions of a printed circuit. In all linear programming problems you need to consider if the variables are integers.



Worked Example 1

Show that $x = 5, y = 3$ satisfies all the constraints.

Find the associated profit for this solution, and compare this profit with other possible solutions.



Solution

For the point $(5, 3)$ we check the constraints:

$$20x + 10y = 130 \leq 200$$

$$10x + 20y = 110 \leq 120$$

$$10x + 30y = 140 \leq 150$$

and clearly, $8 \geq 0$ and $5 \geq 0$. Hence the point $(5, 3)$ satisfies the constraints and the associated profit is

$$\begin{aligned} P &= 5x + 12y \\ &= 25 + 36 \end{aligned}$$

$$\text{i.e. } P = 61$$

We can consider many other points; for example,

(i) $(7, 2)$: this satisfies the constraints and $P = 59$

(ii) $(0, 5)$: this satisfies the constraints and $P = 60$

(iii) $(3, 4)$: this satisfies the constraints and $P = 63$

So the point that satisfied the constraints and gives the maximum profit for the points tested is $(3, 4)$, that is $x = 3$ and $y = 4$.

3.1

At this stage, we will not continue with finding the actual solutions but will concentrate on further practice in formulating problems of this type.

The key stage is the first one, namely that of identifying the unknowns; so you must carefully read the problem through in order to identify the basic unknowns. Once you have done this successfully, it should be straight forward to express both the constraints and the profit function in terms of the unknowns.

A further condition to note in many of the problems is that the unknowns must be positive **integers**.



Worked Example 2

A small firm builds two types of garden shed.

Type A requires 2 hours of machine time and 5 hours of craftsman time.

Type B requires 3 hours of machine time and 5 hours of craftsman time.

Each day there are 30 hours of machine time available and 60 hours of craftsman time. The profit on each type A shed is £60 and on each type B shed is £84.

Formulate the appropriate linear programming problem.



Solution

(a) Unknowns

Define

x = number of Type A sheds produced each day,

y = number of Type B sheds produced each day.

(b) Constraints

Machine time: $2x + 3y \leq 30$

Craftsman time: $5x + 5y \leq 60$

and $x \geq 0, y \geq 0$

(c) Profit

$$P = 60x + 84y$$

So, in summary, the linear programming problem is

maximise $P = 60x + 84y$

subject to $2x + 3y \leq 30$

$$x + y \leq 12$$

$$x \geq 0$$

$$y \geq 0$$

3.1



Exercises

1. Ann and Margaret run a small business in which they work together making blouses and skirts.
 Each blouse takes 1 hour of Ann's time together with 1 hour of Margaret's time. Each skirt involves Ann for 1 hour and Margaret for half an hour. Ann has 7 hours available each day and Margaret has 5 hours each day.
 They could just make blouses or they could just make skirts or they could make some of each.
 Their first thought was to make the same number of each. But they get £8 profit on a blouse and only £6 on a skirt.
 - (a) Formulate the problem as a linear programming problem.
 - (b) Find three solutions which satisfy the constraints.

2. A distribution firm has to transport 1200 packages using large vans which can take 200 packages each and small vans which can take 80 packages each. The cost of running each large van is £40 and of each small van is £20. Not more than £300 is to be spent on the job. The number of large vans must not exceed the number of small vans.
 Formulate this problem as a linear programming problem given that the objective is to **minimise** costs.

3. A firm manufactures wood screws and metal screws. All the screws have to pass through a threading machine and a slotting machine. A box of wood screws requires 3 minutes on the slotting machine and 2 minutes on the threading machine. A box of metal screws requires 2 minutes on the slotting machine and 8 minutes on the threading machine. In a week, each machine is available for 60 hours.
 There is a profit of £10 per box on wood screws and £17 per box on metal screws.
 Formulate this problem as a linear programming problem given that the objective is to **maximise** profit.

4. A factory employs unskilled workers earning £135 per week and skilled workers earning £270 per week. It is required to keep the weekly wage bill below £24 300.
 The machines require a minimum of 110 operators, of whom at least 40 must be skilled. Union regulations require that the number of skilled workers should be at least half the number of unskilled workers.
 If x is the number of unskilled workers and y the number of skilled workers, write down all the constraints to be satisfied by x and y .

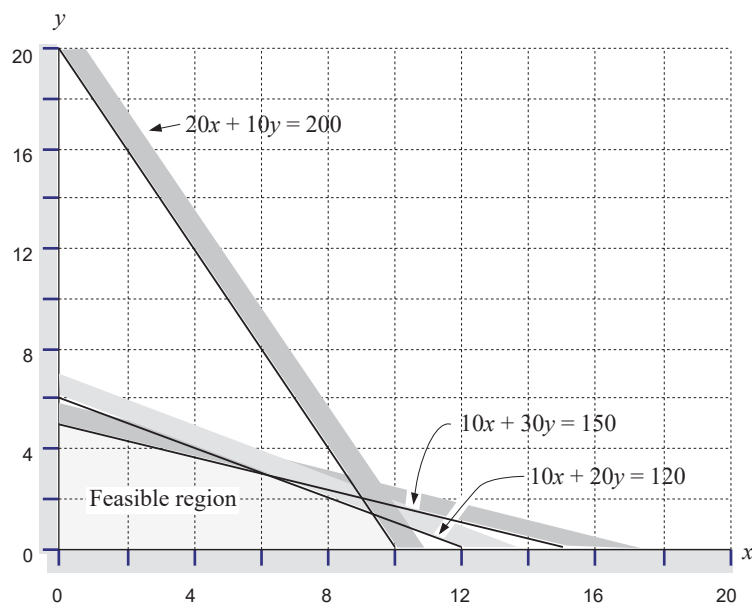
3.2 Graphical Solution

In the previous section you worked through problems that led to a linear programming problem in which a **linear** function of x and y is to be maximised (or minimised) subject to a number of **linear** inequalities to be satisfied.

Fortunately problems of this type with just two variables can easily be solved using a graphical method. The method will first be illustrated using the example from the text in section 3.1. This resulted in the linear programming problem

$$\begin{aligned} &\text{maximise } P = 5x + 12y \\ &\text{subject to } 20x + 10y \leq 200 \\ &\quad 10x + 20y \leq 120 \\ &\quad 10x + 30y \leq 150 \\ &\quad x \geq 0 \\ &\quad y \geq 0 \end{aligned}$$

You can illustrate the **feasible** (i.e. allowable) region by graphing all the inequalities and shading out the regions not allowed. This is illustrated in the figure below.



Magnifying the feasible region, you can look at the family of straight lines defined by

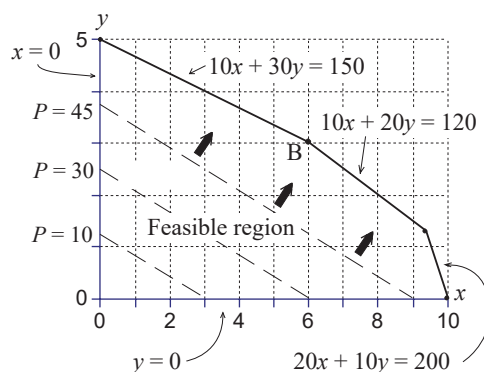
$$P = 5x + 12y$$

where P takes various values.

The figure shows, for example, the lines defined by

$$P = 15, P = 30, \text{ and } P = 45$$

On each of these lines any point gives the same profit.



3.2

**Worked Example 1**

Check that the points

$$x = 1, y = \frac{25}{12}$$

$$x = 2, y = \frac{5}{3}$$

$$x = 4, y = \frac{5}{6}$$

each lie on the line defined by $P = 30$. What profit does each of these points give?

**Solution**

$$P = 5x + 12y$$

$$\left(1, \frac{25}{12}\right), \quad P = 5 \times 1 + 12 \times \frac{25}{12} = 5 + 25 = 30$$

$$\left(2, \frac{5}{3}\right), \quad P = 5 \times 2 + 12 \times \frac{5}{3} = 10 + 20 = 30$$

$$\left(4, \frac{5}{6}\right), \quad P = 5 \times 4 + 12 \times \frac{5}{6} = 20 + 10 = 30$$

So all three points lie on the line defined by $P = 30$ and so each gives a profit of 30.

Note that as the profit line moves to the right, the profit increases and so the maximum profit corresponds to the last point touched as the profit line moves out of the feasible region.

This is the point B, the intersection of

$$10x + 30y = 150 \text{ and } 10x + 20y = 120$$

Solving these equations gives $10y = 30$, i.e. $y = 3$ and $x = 6$. So maximum profit occurs at the point (6, 3) and the profit is given by

$$P = 5 \times 6 + 12 \times 3 = 66$$

**Worked Example 2**

A farmer has 20 hectares for growing barley and swedes. The farmer has to decide how much of each to grow. The cost per hectare for barley is £30 and for swedes is £20. The farmer has budgeted £480.

Barley requires 1 man-day per hectare and swedes require 2 man-days per hectare. There are 36 man-days available.

The profit on barley is £100 per hectare and on swedes is £120 per hectare.

Find the number of hectares of each crop the farmer should sow to maximise profits.

3.2



Solution

The problem is formulated as a linear programming problem:

(a) **Unknowns**

x = number of hectares of barley

y = number of hectares of swedes

(b) **Constraints**

Land $x + y \leq 20$

Cost $30x + 20y \leq 480$

Manpower $x + 2y \leq 36$

(c) **Profit**

$$P = 100x + 120y$$

To summarise, maximise $P = 100x + 120y$

subject to $x + y \leq 20$

$30x + 20y \leq 480$

$x + 2y \leq 36$

$x \geq 0$

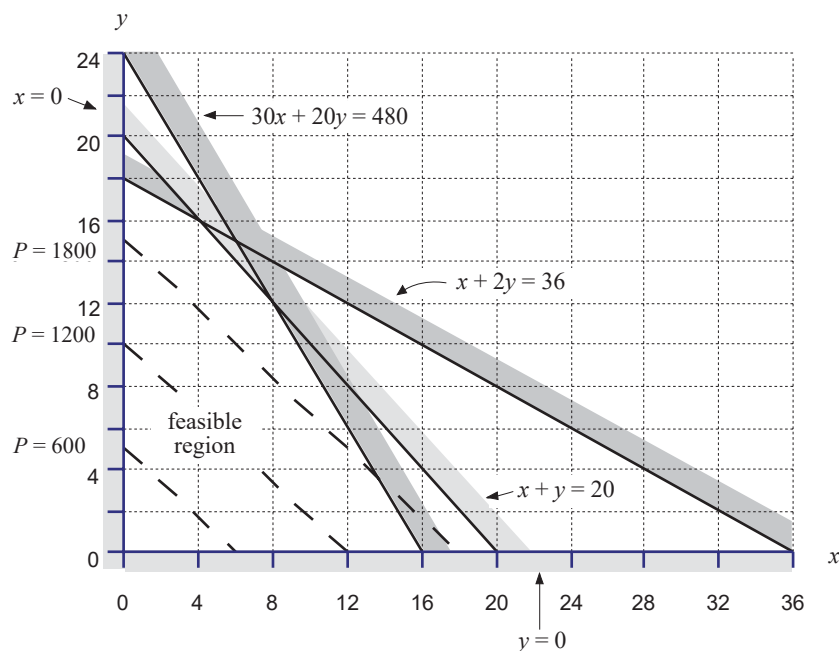
$y \geq 0$

The feasible region is identified by the region enclosed by the five inequalities, as shown below. The profit lines are given by

$$P = 100x + 120y$$

and again you can see that P increases as the line (shown dotted) moves to the right. Continuing in this way, the maximum profit will occur at the intersection of

$$x + 2y = 36 \text{ and } x + y = 20$$



3.2

At this point $x = 4$ and $y = 16$ and the corresponding maximum profit is given by

$$P = 100 \times 4 + 120 \times 16 = 2320$$

The farmer should sow 4 hectares with barley and 16 with swedes.



Exercises

- Solve the linear programming problem defined in Question 1 of the Exercises in Section 3.1.
- Solve the linear programming problem defined in Question 2 of the Exercises in Section 3.1.
- A camp site for caravans and tents has an area of 1800 m^2 and is subject to the following regulations:
The number of caravans must not exceed 6.
Reckoning on 4 persons per caravan and 3 per tent, the total number of persons must not exceed 48.
At least 200 m^2 must be available for each caravan and 90 m^2 for each tent.
The nightly charges are £2 for a caravan and £1 for a tent.
(a) Find the greatest possible nightly takings.
How many caravans and tents should be admitted if the site owner wants to make the maximum profit and have
(b) (i) as many caravans as possible,
(b) (ii) as many tents as possible?
- The annual subscription for a tennis club is £20 for adults and £8 for juniors. The club needs to raise at least £800 in subscriptions to cover its expenses.
The total number of members is restricted to 50. The number of junior members is to be between one quarter and one third of the number of adult members.
(a) (i) Represent the information graphically,
(ii) Find the numbers of adult and junior members which will bring in the largest amount of money in subscriptions.
(b) Find the least total membership which will satisfy the conditions.
- The numbers of units of vitamins A, B and C in a kilogram of foods X and Y are as follows:

Food	Vitamin A	Vitamin B	Vitamin C
X	5	2	6
Y	4	6	2

A mixture of the two foods is made which has to contain at least 20 units of vitamin A, at least 24 units of vitamin B and at least 12 units of vitamin C.

- Find the smallest total amount of X and Y to satisfy these constraints.

3.2

Food Y is three times as expensive as Food X.

(b) Find the amounts of each to minimise the cost and satisfy the constraints.

6. A firm manufactures two types of box, each requiring the same amount of material.

They both go through a folding machine and a stapling machine.

Type A boxes require 4 seconds on the folding machine and 3 seconds on the stapling machine.

Type B boxes require 2 seconds on the folding machine and 7 seconds on the stapling machine.

Each machine is available for 1 hour.

There is a profit of 40p on Type A boxes and 30p on Type B boxes.

How many of each type should be made to maximise the profit?

7. A small firm which produces radios employs both skilled workers and apprentices. Its workforce must not exceed 30 people and it must make at least 360 radios per week to satisfy demand. On average a skilled worker can assemble 24 radios and an apprentice 10 radios per week.

Union regulations state that the number of apprentices must be less than the number of skilled workers but more than half of the number of skilled workers.

What is the greatest number of skilled workers than can be employed?

Skilled workers are paid £300 a week, and apprentices £100 a week.

How many of each should be employed to keep the wage bill as low as possible?

8. In laying out a car park it is decided, in the hope of making the best use of the available parking space (7200 sq.ft.), to have some spaces for small cars, the rest for large cars. For each small space 90 sq.ft. is allowed, for each large space 120 sq.ft. Every car must occupy a space of the appropriate size. It is reliably estimated that, of the cars wishing to park at any given time, the ratio of small to large will be neither less than 2:3 nor greater than 2:1.

Find the number of spaces of each type in order to maximise the number of cars that can be parked.

9. A contractor hiring earth moving equipment has the choice of two machines.

Type A costs £25 per day to hire, needs one man to operate it and moves 30 tonnes of earth per day.

Type B costs £10 per day to hire, needs four men to operate it and moves 70 tonnes of earth per day.

The contractor can spend up to £500 per day, has a labour force of 64 men available and can use a maximum of 25 machines on the site.

Find the maximum weight of earth that the contractor can move in one day.

10. A landscape designer has £200 to spend on planting trees and shrubs to landscape an area of 1000 m². For a tree he plans to allow 25 m² and for a shrub 10 m². Planting a tree will cost £2 and a shrub £5.

3.2

If he plants 30 shrubs what is the maximum number of trees he can plant?

If he plants 3 shrubs for every tree, what is the maximum number of trees he can plant?

11. A small mine works two coal seams and produces three grades of coal. It costs £10 an hour to work the upper seam, obtaining in that time 1 tonne of anthracite, 5 tonnes of best quality coal and 2 tonnes of ordinary coal. The lower seam is more expensive to work, at a cost of £15 per hour, but it yields in that time 4 tonnes of anthracite, 6 tonnes of best coal and 1 tonne of ordinary coal. Faced with just one order, for 8 tonnes of anthracite, 30 tonnes of best coal and 8 tonnes of ordinary coal, how many hours should each seam be worked so as to fill this order as cheaply as possible?

3.3 Simplex Method

Looking back at Worked Example 2 in Section 3.2, the slope of the profit line was $\frac{5}{6}$. (The slope is actually negative but it is sufficient to just consider the magnitude of the slopes of the lines.)

This is more than the slope of the line $x + 2y = 36$ (namely $\frac{1}{2}$), but less than the slope of the other two lines, $x + y = 20$ (i.e. 1) and $30x + 20y = 480$ (i.e. $\frac{3}{2}$).

So the solution will occur at the intersection of the two lines with slopes $\frac{1}{2}$ and 1.

**Worked Example 1**

Check the slopes of the constraints and profit function in the first example in the text in Section 3.2.

**Solution**

The slopes are $20x + 10y = 200 \Rightarrow 2x + y = 20 \Rightarrow 2$

$$10x + 20y = 120 \Rightarrow \frac{1}{2}x + y = 6 \Rightarrow \frac{1}{2}$$

$$10x + 30y = 150 \Rightarrow \frac{1}{3}x + y = 5 \Rightarrow \frac{1}{3}$$

whilst the profit function $P = 5x + 12y$ has slope $\frac{5}{12}$.

This lies between $\frac{1}{3}$ and $\frac{1}{2}$ and so the maximum profit is at the intersection of

$$\left. \begin{array}{l} 10x + 20y = 120 \\ 10x + 30y = 150 \end{array} \right\} x = 6, y = 3$$

with profit $P = 5 \times 6 + 12 \times 3 = 66$.

3.3

Another point worth noting here is that the solution of a linear programming problem will occur, in general, at one of the **vertices** of the feasible region if either non-integer solutions are acceptable or the vertex in question happens to have integral coordinates.

So an alternative to the graphical method of solution would be to

- find all the vertices of the feasible region;
- find the value of the profit function at each of these vertices;
- choose the one which gives maximum value to the profit function.

(We have not considered the possibility of non-integer vertices, which may not make sense in terms of the original problem.)

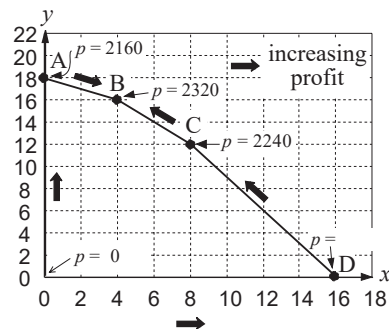
You can see how this method works with the second Worked Example in Section 3.2.

The vertices are given by

- $O(0, 0)$
- $A(0, 18)$
- $B(4, 16)$
- $C(8, 12)$
- $D(16, 0)$

and the corresponding profits in £ are

Point	Profit
O	0
A	2160
B	2320
C	2240
D	1600



As you can see, as you move round the feasible region, the profit increases from 0 to A to B, but then decreases to C to D and back to 0.

In more complicated problems, it is helpful to introduce the idea of **slack variables**. For the problem above, with the three inequalities

$$x + y \leq 20$$

$$30x + 20y \leq 480$$

$$x + 2y \leq 36$$

three new variables are defined by

$$r = 20 - x - y$$

$$s = 480 - 30x - 20y$$

$$t = 36 - x - 2y$$

The three inequalities can now be written as

$$r \geq 0, s \geq 0, t \geq 0$$

3.3

as well as $x \geq 0, y \geq 0$. The variables r, s and t are called the slack variables as they represent the amount of slack between the total quantity available and how much is being used.

The importance of the slack variables is that you can now define each vertex in terms of two of the variables, x, y, r, s , or t , being zero; for example, at A,

$$x = t = 0$$



Worked Example 2

Complete the table below, defining each vertex

$$0 \quad x = y = 0$$

$$A \quad t = x = 0$$

$$B \quad \dots\dots\dots$$

$$C \quad \dots\dots\dots$$

$$D \quad \dots\dots\dots$$



Solution

$$B : \quad r = t = 0$$

$$C : \quad s = r = 0$$

$$D : \quad y = s = 0$$

The procedure of increasing the profit from one vertex to the next will be again followed. Starting at the origin

$$P = 100x + 120y = 0 \text{ at } x = y = 0$$

Clearly P will increase in either direction from the origin - moving up the y axis means that x is held at zero whilst y increases. Although you know from the diagram that the next vertex reached will be A, how could you work that out without a picture? As x is being kept at zero and y is increasing the next vertex met will either be where $x = r = 0$ or where $x = s = 0$ or where $x = t = 0$. But note that

$$x = r = 0 \Rightarrow y = 20$$

$$x = s = 0 \Rightarrow y = 24$$

$$x = t = 0 \Rightarrow y = 18 \leftarrow \text{smallest}$$

and so as y increases the first of these three points it reaches is the one where y is the smallest, namely $y = 18$ and the vertex is where $x = t = 0$ (which is the point A, thus confirming without a picture what has already been seen). You can now express the profit P in terms of x and t :

$$\begin{aligned} P &= 100x + 120y \\ &= 100x + 120 \frac{(36 - x - t)}{2} \\ &= 40x - 60t + 2160 \end{aligned}$$

You can continue to increase P as the coefficient of x is positive.

3.3

So you now increase x , keeping t at zero. You will next meet a vertex where either $t = s = 0$, $t = r = 0$ or $t = y = 0$. As x is increasing the vertex will be the one of those three points where x is the smallest:

$$t = s = 0 \Rightarrow x = 6$$

$$t = r = 0 \Rightarrow x = 4 \leftarrow \text{smallest}$$

$$t = y = 0 \Rightarrow x = 36$$

The next vertex reached is where $t = r = 0$ (which is B) and P is now expressed in terms of t and r . To do this eliminate y from the first and third of the original equations by noting that

$$2r - t = 2(20 - x - y) - (36 - x - 2y) = 4 - x$$

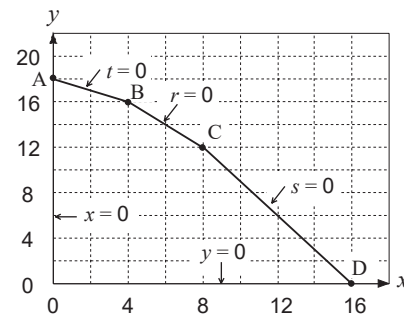
Hence

$$P = 40x - 60t + 2160$$

$$= 40(4 - 2r + t) - 60t + 2160$$

$$= -80r - 20t + 2320$$

Throughout the feasible region you know that $r \geq 0$ and $t \geq 0$ and so it is clear from the negative coefficients in the above expression for P that P reaches its maximum value of 2320 when r and t are both zero. This happens when $x = 4$ and $y = 16$.



Exercises

1. (a) Solve the linear programming problem maximise $P = 2x + 4y$
subject to $x + 5y \leq 10$
$$4x + y \leq 8$$
$$x \geq 0$$
$$y \geq 0$$
by a graphical method.
- (b) Introduce slack variables r and s , and solve the problem by the simplex method.
2. (a) Determine the vertices of the feasible region for the linear programming problem
maximise $P = x + y$
subject to $x + 4y \leq 8$
$$2x + 3y \leq 12$$
$$3x + y \leq 9$$
$$x \geq 0$$
$$y \geq 0$$
Hence find the solution.

3.3

- (b) Verify this solution by using the simplex method.
3. Use the simplex method to solve the linear programming problem
- maximise $P = 10x + 15y$
- subject to $4y + 10x \leq 40$
- $10y + 3x \leq 30$
- $5y + 4x \leq 20$
- $x \geq 0$
- $y \geq 0$