

STRAND: PROBLEM SOLVING

Unit 1 *Mathematical Modelling*

TEXT

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1 Mathematical Modelling

1.1 Mathematical Modelling

Mathematical modelling is central to all the CORE MATHS syllabi so we will start by looking at a number of contrasting case studies illustrating how mathematics is used to solve problems. We will show, in particular, how mathematics is used

- to explain
- to predict
- to make decisions

and this will be illustrated in the first section.

A mathematical model is a description of a system using mathematical concepts and notation; the process of developing a mathematical model is called 'mathematical modelling'. A variety of mathematical models is illustrated in the following case studies.

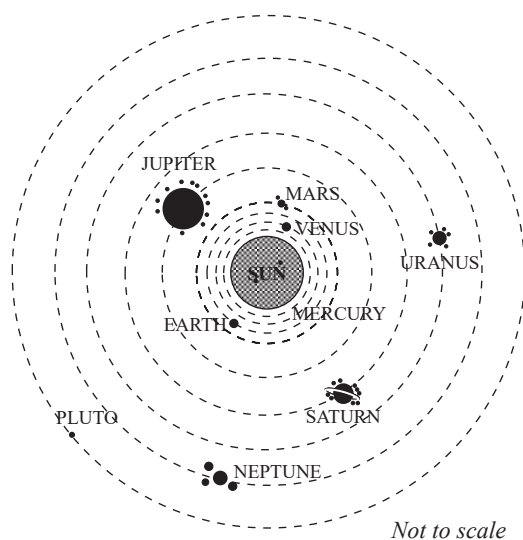
Bode's law

In 1772, the German astronomer, *Johann Bode*, investigated the pattern formed by the distances of planets from the sun.

At the time, only six planets were known, and the pattern he devised is shown in the table below. The distances are measured on a scale that equates 10 units to the Sun - Earth distance.

The fit between actual distances and Bode's pattern is remarkably good.

Planet	Actual distance	Bode's pattern
Mercury	4	$0 + 4 = 4$
Venus	7	$3 + 4 = 7$
Earth	10	$6 + 4 = 10$
Mars	15	$12 + 4 = 16$
-	-	-
Jupiter	52	$48 + 4 = 52$
Saturn	96	$96 + 4 = 100$



1.1

The pattern can be written as a mathematical model:

$$\text{Distance} = 4 + 3 \times 2^n$$

where

$n = 0$ represents Venus

$n = 1$ represents Earth

$n = 2$ represents Mars

$n = 3$ represents ?

$n = 4$ represents Jupiter

$n = 5$ represents Saturn

There are also planets further out than Saturn (see Exercises 1.1, Question 1).

Wind chill

When the temperature drops to near zero, it is usual for weather forecasters to give both the expected air temperature, and the **wind chill** temperature - this is the temperature actually felt by someone, which depends on the wind speed and air temperature. So, for example, the wind chill temperature for an actual temperature of 0°C and wind speed of 10 mph is given by -5.5°C . For $v > 5$ mph, the wind chill temperature is given by

$$T = 33 + (0.45 + 0.29\sqrt{v} - 0.02v)(t - 33)$$

where $t^\circ\text{C}$ is the air temperature and v mph the wind speed. This formula was devised by American scientists during the Second World War, and is based on experimental evidence.



Worked Example 1

Find the wind chill temperature when

(a) $t = 2^\circ\text{C}$, $v = 20$ mph; (b) $t = -10^\circ\text{C}$, $v = 5$ mph;

(c) $t = 0^\circ\text{C}$, $v = 40$ mph.



Solution

(a) When $t = 2$, $v = 20$,

$$\begin{aligned} T &= 33 + (0.45 + 0.29\sqrt{20} - 0.02 \times 20)(-31) \\ &= -8.8^\circ\text{C} \end{aligned}$$

(b) When $t = -10$, $v = 5$,

$$\begin{aligned} T &= 33 + (0.45 + 0.29\sqrt{5} - 0.02 \times 5)(-43) \\ &= -9.9^\circ\text{C} \end{aligned}$$

1.1

(c) When $t = 0^\circ\text{C}$, $v = 40$ mph,

$$\begin{aligned} T &= 33 + (0.45 + 0.29\sqrt{40} - 0.02 \times 40)(-33) \\ &= -16.0^\circ\text{C} \end{aligned}$$

Heptathlon

The Heptathlon is a competition for female athletes who take part in **seven** separate events (usually spread over a two-day period). For each event, there is a point scoring system, based on the idea that a good competitor will score 1000 points in each event. For example, the points scoring system for the 800 m running event is

$$P = 0.11193 (254 - t)^{1.88}$$

where t is the time taken in seconds for the athlete to run 800 m.



Worked Example 2

What points are scored for a time of 124.2 seconds, and what time would give a point score of 1000?



Solution

For $t = 124.2$,

$$\begin{aligned} P &= 0.11193 (254 - 124.2)^{1.88} \\ \Rightarrow P &= 1051 \end{aligned}$$

(Scores are always rounded down to the nearest whole number.)

Now, to score 1000 points requires a time of m seconds where

$$\begin{aligned} 1000 &= 0.11193 (254 - t)^{1.88} \\ \Rightarrow (254 - t)^{1.88} &= 8934.15 \\ \Rightarrow 254 - t &= (8934.15)^{\frac{1}{1.88}} \\ \Rightarrow t &= 254 - 126.364 \end{aligned}$$

giving $t = 127.64$

All track events use a points scoring system of the form

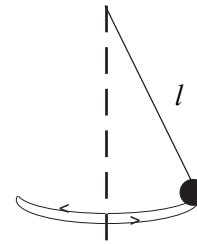
$$P = a(b - t)^c$$

where t is the time taken, with suitable constants a , b and c .

1.1

Simple pendulum

The great Italian scientist, *Galileo*, was the first to make important discoveries about the behaviour of swinging weights. These discoveries led to the development of pendulum clocks. You can easily deduce Galileo's result from a simple experiment, as follows.



Attach a weight at one end of a light string, the other end being fixed.

Let the pendulum swing freely in a vertical plane and for various lengths of pendulum, l , in metres, find the corresponding times in seconds of one complete oscillation (known as the period) - it is more accurate to time, say, five oscillations and then divide the total time by 5. On a graph, plot the period, T , against the square root of the pendulum length, $\ell^{\frac{1}{2}}$. What do you notice?

In fact, the two quantities are related by the formula

$$T = 2.006 \ell^{\frac{1}{2}}$$



Worked Example 3

What pendulum length gives a periodic time of 1 second?



Solution

If $T = 1$ then

$$\begin{aligned} 1 &= 2.006 \ell^{\frac{1}{2}} \\ \Rightarrow \ell^{\frac{1}{2}} &= \frac{1}{2.006} = 0.4985 \\ \Rightarrow \ell &\approx 0.25 \text{ m} \end{aligned}$$

Perfect numbers

These are numbers whose divisors (excluding the number itself) add up to the number. Excluding the number 1, the first perfect number is 6, since

$$6 = 3 \times 2 = 1 \times 6$$

and $3 + 2 + 1 = 6$

Test the numbers 7, 8, ..., 30 to see if you can find the next perfect number.

You have probably realised by now that perfect numbers are pretty thin on the ground!



Worked Example 4

Are the following numbers perfect :

- (a) 220 (b) 284 (c) 496 ?

1.1

**Solution**

$$(a) \quad 220 = 220 \times 1$$

$$= 110 \times 2$$

$$= 55 \times 4$$

$$= 44 \times 5$$

$$= 22 \times 10$$

$$= 20 \times 11$$

$$\text{and } 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$$

Hence 220 is not a perfect number.

$$(b) \quad 284 = 284 \times 1$$

$$= 142 \times 2$$

$$= 71 \times 4$$

$$\text{and } 1 + 2 + 4 + 71 + 142 = 220$$

Hence 284 is not a perfect number (but note its connection with 220).

$$(c) \quad 496 = 496 \times 1$$

$$= 248 \times 2$$

$$= 124 \times 4$$

$$= 62 \times 8$$

$$= 31 \times 16$$

$$\text{and } 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 = 496$$

Hence 496 is a perfect number.

In fact 496 is the third perfect number, and 8128 is the fourth. Although there are still many unknown results concerning perfect numbers, it has been shown that

- (a) all **even** perfect numbers will be of the form

$$2^{n-1}(2^n - 1)$$

when n is a prime number. This number is in fact perfect when $2^n - 1$ is prime;

- (b) all even perfect numbers end in 6 or 8;

- (c) the sum of the inverses of all divisors of a perfect number add up to 2,

$$\text{e.g. for } 6, \quad \frac{1}{6} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} = 2$$

You probably noticed in the example above that 220 and 284 are connected through their divisors. They are called **amicable pairs** (they are the smallest numbers that exhibit this property) and are regarded as tokens of great love. In the Bible, for example, Jacob gave Esau 220 goats to express his love (Genesis Ch. 32, verse 14). You might consider writing a short program to generate amicable pairs, and use it to find the next lowest pair.

1.1

This is not quite a mathematical model but rather a way of generating perfect numbers. As in some other areas in mathematics, the applications come later, with very large numbers, such as perfect numbers, being important for internet security.

Day of the week

The algorithm below gives a method for determining the day of the week for any date this century. The date used as an example is 3 March, 2015.

- | | |
|---|--|
| 1. Write $y = \text{year}$ | $y = 2015$ |
| 2. Evaluate $\left\lfloor \frac{y-1}{4} \right\rfloor$
ignoring the remainder | $\left\lfloor \frac{y-1}{4} \right\rfloor = \left\lfloor \frac{2014}{4} \right\rfloor = 503$ |
| 3. Find $D = \text{day of year}$.
(Jan. 1st = 1, ... ,
Feb. 1st = 32, etc.) | $D = 31 + 28 + 3 = 62$ |
| 4. Calculate

$s = y + \left\lfloor \frac{y-1}{4} \right\rfloor + D$ | $s = 2015 + 503 + 62$

$= 2580$ |
| 5. Divide by 7 and note
remainder, R | $\frac{s}{7} = \frac{2580}{7} = 368$
with remainder $R = 4$ |
| 6. The remainder is the key
to the day :

$R = 0 \Rightarrow \text{Friday}$
$R = 1 \Rightarrow \text{Saturday, etc.}$ | Hence 3 March 2015 is
a Tuesday. |

These case studies illustrate a range of mathematical models. The first one, *Bode's law*,

$$\text{distance} = 4 + 3 \times 2^n$$

is a formula, based on known data but not on any theoretical model.

This is also true for the second case study, *wind chill*, which is used extensively by climbers and explorers in very cold climates!

The third mathematical model is a formula designed to give a positive score of 1000 for a world class athlete completing the 800 m running event in the *heptathlon* and a score of zero for a time of just over 4 minutes.

1.1

In contrast, the time of a *pendulum swing* is based on a theoretical underpinning, namely Newton's law of motion, whilst our next problem, that of the distribution of *perfect numbers*, is a suggested formula to find such numbers, that is

$$2^{n-1}(2^n - 1)$$

when n is a prime number. When $2^n - 1$ is also prime, this formula generates a perfect number.



Exercises

1. **Bode's law** ($d = 4 + 3 \times 2^n$)

What are the next three numbers ($n = 6, 7, 8$) in Bode's formula? More planetary data is shown below.

Planet	Actual distance
Uranus	192
Neptune	301
Pluto	395

Does this extra data provide further evidence for Bode's law?

How could $n = 3$ be interpreted? What is the value of n that would correspond to the planet Mercury?

2. **Wind chill** ($T = 33 + (0.45 + 0.29\sqrt{v} - 0.02v)(t - 33)$)

(a) Use the wind chill temperature formula to find its value (T) where

(i) $t = 0^\circ\text{C}$, $v = 20$ mph

(ii) $t = 5^\circ\text{C}$, $v = 20$ mph

(iii) $t = -5^\circ\text{C}$, $v = 20$ mph

Plot a graph of wind chill temperature against air temperature, t , for $v = 20$ mph. Use your graph to estimate the wind chill temperature when $t = 10^\circ\text{C}$ and $v = 20$ mph.

(b) What happens as v gets larger?

(c) For what speeds does the formula predict that the actual temperature ($t^\circ\text{C}$) is equal to the wind chill temperature ($T^\circ\text{C}$)?

3. **Heptathlon** ($P = 0.11193(254 - t)^{1.88}$)

(a) For the 800 m points scoring formula, what time scores zero points?

(b) Find the points scored for the current 800 World Record.

(c) How could you adapt the points formula for running events to be used for field events (high jump, long jump, discus and javelin)?

1.1

4. **Simple pendulum** ($T = 2.006 \ell^{\frac{1}{2}}$)

Construct a simple pendulum with $\ell = 0.25$ m and check its periodic time with the formula.

5. **Perfect numbers**

Given that the fifth and sixth perfect numbers are 33 550 336 and 8 589 869 056 respectively. Copy and complete the table below.

n	$2^n - 1$	prime	$2^{n-1}(2^n - 1)$	perfect
2	3	✓	6	✓
3				
5				
7				
11				
13				

6. **Day of the week**

- (a) Use the algorithm to find the day of the week on which you were born. Note that if the year is divisible by 4, it is a leap year and February has 29 days!
- (b) Will this algorithm work for dates in the last century?

1.2 Using and Applying Mathematics

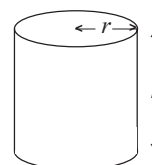
Mathematics can be a very powerful tool in solving practical problems. An example of this is given below with an optimisation problem of the type met in the commercial world, as well as two further case studies showing how mathematics is used to solve problems.

Metal cans

The most popular size of metal can contains a volume of about 440 ml. As they are produced in millions each week, any savings that can be made in their manufacture will prove significant. Part of the cost of making steel cans is based on the amount of metal used, so it might be sensible to design a can which minimises the amount of metal needed to enclose the required volume.

To analyse this problem, you must find an expression for the total surface area of a can. Suppose the cylindrical can has radius r and height h , then total surface area,

$$S = \text{curved surface area} + \text{top area} + \text{base area}.$$



1.2

Assuming that no metal is wasted, an expression for the total surface area is given by

$$S = 2\pi rh + \pi r^2 + \pi r^2$$

$$\Rightarrow S = 2\pi rh + 2\pi r^2 \quad (1)$$

The formula for S shows that it is a function of two variables, r and h . But in reality it is a function of only one variable since r and h are constrained by having to enclose a specified volume.

You should be familiar with the formula for the volume of a cylindrical can:

$$V = \text{area of cross section} \times \text{height}$$

or, in this case

$$440 = \pi r^2 h \quad (2)$$

This equation can be used to find an expression for h which is substituted into (1) to eliminate h .

From (2)

$$h = \frac{440}{\pi r^2} \quad (3)$$

and substituting into (1) gives

$$S = 2\pi r \left(\frac{440}{\pi r^2} \right) + 2\pi r^2$$

giving

$$S = \frac{880}{r} + 2\pi r^2 \quad (4)$$

The problem is to find the value of r which minimises the total surface area S .



Worked Example 1

Draw the graph of S against r for $r = 2, \dots, 8$ and hence determine an approximation for the optimal value of r and corresponding value of h . (Alternatively use a graph calculator or spreadsheet.)



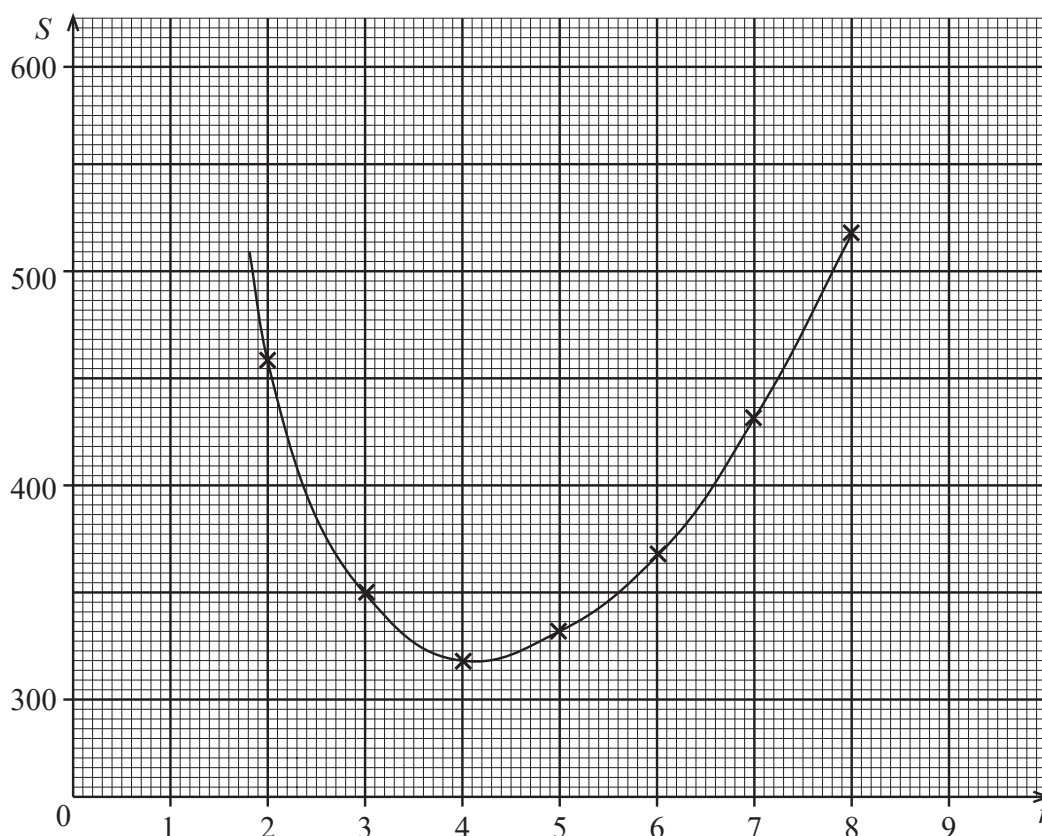
Solution

The table of values is

r	2	3	4	5	6	7	8
S	465.1	349.9	320.5	333.1	372.9	433.6	512.1

and this is shown below on the following graph for S against r :

1.2



You can see from the graph that S is minimised where r is approximately 4.1 cm and, from equation (2) above,

$$h = \frac{440}{\pi r^2} = \frac{440}{\pi (4.1)^2} \approx 8.3 \text{ cm}$$

(note that h is approximately $2 \times r$)

The next stage is to check whether $h \approx 2 \times r$ (= diameter) is in fact used in practice (see Q1 in Exercise 1.2).

Reading age formula

Educationalists need to be able to assess the minimum reading age of certain books so that they can be appropriately catalogued, particularly for use with young children.

You are probably aware that, for example, it is much easier and quicker to read one of the tabloids (e.g. 'The Sun') than one of the quality 'heavies' (e.g. 'The Guardian'). What factors influence the reading age of a book, newspaper or pamphlet?

There have been many attempts at designing a formula for finding the reading age of a text. One example is known as the **FOG** index. This is given by

$$R = \frac{2}{5} \left(\frac{A}{n} + \frac{100L}{A} \right) \quad (5)$$

where the variables are defined for a sample passage of the text by

1.2

A = number of words

n = number of sentences

L = number of words containing three or more syllables (excluding '-ing' and '-ed' endings).



Worked Example 2

Here is a passage from the book *Buried Alive* by Jacqueline Wilson.

The moment we got to the caravan site and saw the ropes and flags set out across the beach I realised something terrible.

There was going to be sports.

I am the least sporty boy ever.

'Great!' said Dad, reading the poster. 'There's going to be all sorts of races. Sprinting, relay, three-legged, sack-race, egg and spoon. You boys must have a go.'

'It'll be just for people staying at the caravan site,' I said quickly.

'We can't enter, it wouldn't be fair.'

'Don't be such a wimp, Tim,' Dad said sharply. 'Of course you can enter.'

'But I don't want to!' I said.

'Nor do I, actually,' said Biscuits loyally.

'There! We'd have all been much better off if we'd gone for a car ride,' said Mum. 'In fact, why don't we still go? This carnival doesn't look very exciting. There aren't any craft or bric-a-brac stalls, and the tombola prizes don't look much cop. There aren't even many food stalls.'

Use the FOG index to estimate the reading age of this book, based on this passage.



Solution

$$R = \frac{2}{5} \left(\frac{A}{n} + \frac{100L}{A} \right)$$

where A = number of words in the passage

n = number of sentences

L = number of words containing 3 or more syllables.

So $A = 167$, $n = 10$, $L = 8$, giving

$$R = \frac{2}{5} \left(\frac{167}{10} + \frac{100 \times 8}{167} \right)$$

$$R \approx 5.6$$

(This may be lower than you might expect but remember it is only a simple formula and will not give an 'exact' answer.)

1.2

Bar code design

Nearly all grocery (and most other) products now include an identifying *bar code* on their wrapper (supermarkets use them both for sales checkout and stock control). There are two types of EAN (the abbreviation for European Article Number although the name has now been changed to International Article Number) – EAN-13 digit and EAN-8 digit. The shortened 8 digit code will be considered here.

A possible example is shown below. The number has three parts:

0 0	3 9 9 1 0	4
↑	↑	↑
retailer's code	product code	check digit

The check digit is chosen so that

$$3 \times (1^{\text{st}} + 3^{\text{rd}} + 5^{\text{th}} + 7^{\text{th}} \text{ numbers}) + (2^{\text{nd}} + 4^{\text{th}} + 6^{\text{th}} + 8^{\text{th}} \text{ numbers})$$

is exactly divisible by 10. For the numbers above

$$\begin{aligned} &3 \times (0 + 3 + 9 + 0) + (0 + 9 + 1 + 4) \\ &= 3 \times 12 + 14 = 36 + 14 = 50 \end{aligned}$$

which is divisible by 10.

If the check digit is in error, the optical bar code reader will reject the code.



Worked Example 3

Find the check digit for the EAN codes :

- (a) 5021421 x (b) 0042655 x



Solution

- (a) Denoting the check digit by x , the number

$$3 \times (5 + 2 + 4 + 1) + (0 + 1 + 2 + x) = 3 \times 12 + 3 + x = 39 + x$$

must be divisible by 10, so x must be 1.

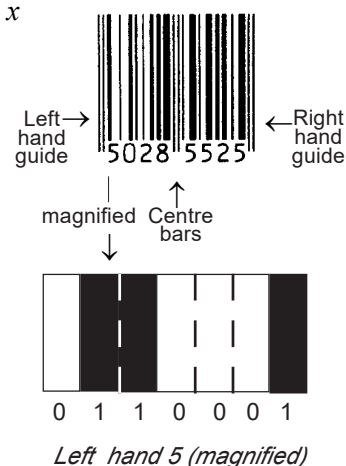
- (b) Similarly

$$3 \times (0 + 4 + 6 + 5) + (0 + 2 + 5 + x) = 3 \times 15 + 7 + x = 52 + x$$

must be divisible by 10, so x must be 8.

An EAN-8 digit bar code is shown opposite. It has left and right hand guide bars and centre bars. In between there are 8 bars of varying thickness. Each number is represented by a unique set of 2 bars and 2 spaces. As can be seen in the magnified version of 5, each number code is made up of 7 **modules**.

The digit 5 is written as 0110001 to indicate whether a module is white (0) or black (1).



1.2

All left hand numbers start with 0 and end with 1, and use a total of 3 or 5 black modules and there must be 3 changes of number, that is, from 0 to 1, 1 to 0 and 0 to 1. This ensures that there are 2 white and two black strips (of varying thickness). Right hand numbers are the reflection of the corresponding left hand code, e.g. right hand 5 is 1000110.



Worked Example 4

Design all possible codes for left hand numbers, using the rules above, that is, using 7 modules and

- starting with 0 and ending with 1
- using a total of 3 or 5 black modules
- changing from 0 to 1, 1 to 0 and 0 to 1.



Solution

Here are possibilities using exactly 3 black modules:

0 1 1 0 0 0 1 (this is a left hand side 5)
 0 0 1 1 0 0 1
 0 0 0 1 1 0 1

0 1 0 0 0 1 1
 0 0 1 0 0 1 1
 0 0 0 1 0 1 1

and using exactly 5 black modules:

0 1 1 1 1 0 1
 0 1 1 1 0 1 1
 0 1 1 0 1 1 1
 0 1 0 1 1 1 1



Note

This gives 10 possibilities, exactly the number needed to code the digits 0 to 9 (see final page of this Text).

Because mathematics is a precise science, applications to real problems require both an understanding of the problem and an appreciation that, whilst mathematics can provide answers and give precise explanations based on particular assumptions and models, it cannot always solve the real problem. Mathematical modelling can help to design multi-stage rockets that work, but it can't necessarily help to solve the problem of world peace. Often mathematical modelling can help in making the best decisions, and, for example, success is shown by the fact that man has stepped on the moon. You should, though, be aware that most problems in real life are more complicated than a single equation or formula!

1.2



Exercises

1. **Metal cans**

Repeat the analysis for finding the minimum surface area of a metal can where the volume enclosed is 1000 ml. Determine the values of r and h which minimise the surface area.

2. Check the dimensions of a range of 440 ml cans. Can you explain why the observed values are not in line with the theoretical values?

Check the dimensions of catering tins that contain large volumes. Does this help provide an explanation?

3. **FOG index** $\left(R = \frac{2}{5} \left(\frac{A}{n} + \frac{100L}{A} \right)\right)$

Use the FOG index to estimate the reading age of the passage below, taken from *The Constant Gardener* by John le Carré.

Free also of his interrogation by the police, when a Justin he didn't recognise strode to the centre of the stage and, in a series of immaculately sculpted sentences, laid his burden at the feet of his bemused interrogators - or as much of it as a puzzled instinct told him it was prudent to reveal. They began by accusing him of murder.

'There's a scenario hanging over us here, Justin,' Lesley explains apologetically, 'and we have to put it to you straight away, so that you're aware of it, although we know it's hurtful. It's called a love triangle, and you're the jealous husband and you've organised a contract killing while your wife and her lover are as far way from you as possible, which is always good for the alibi. You had them both killed, which was what you wanted for your vengeance. You had Arnold Bluhm's body taken out of the jeep and lost so that we'd think Arnold Bluhm was the killer and not you. Lake Turkana's full of crocodiles, so losing Arnold wouldn't be a problem. Plus there's a nice inheritance coming your way by all accounts, which doubles up the motive.'

4. A mathematical model for the reading age of a text is given by

$$R = 25 - \frac{N}{10}$$

where N is the average number of one syllable words in a passage of 150 words. Use this model to find the reading age of a number of books. Compare the results with the FOG index. Is there agreement?

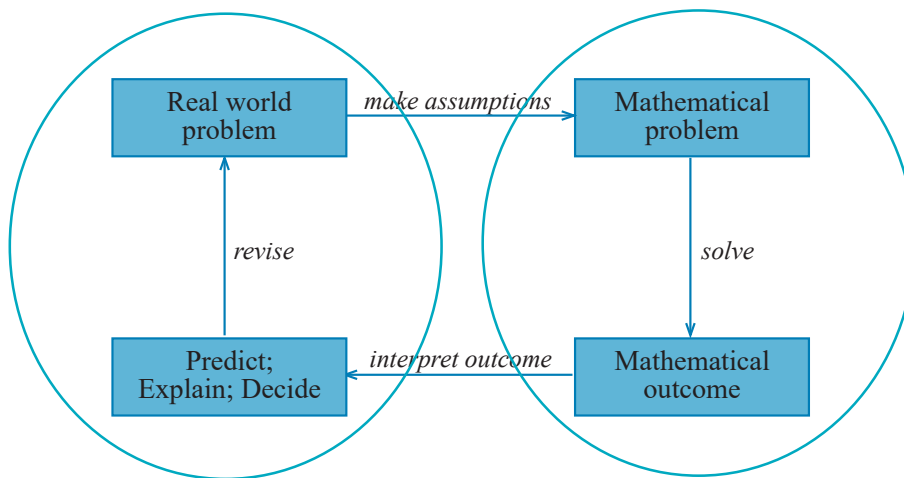
5. **EAN codes** Find the check digits, x , for these EAN-8 digit codes :

- (a) 0034548 x (b) 5023122 x

6. Determine whether these EAN-8 digit codes have the correct check digit :

- (a) 00306678 (b) 06799205

1.3 Summary



Mathematical models have been used in the natural sciences (such as physics, biology, chemistry) and engineering disciplines for many centuries but they are now increasingly used in a wide variety of problems in social sciences. A mathematical model can help to explain a phenomenon or make predictions. The diagram below gives an overview of the mathematical modelling cycle.

Note the phases shown above: you start with a real world problem, making clear what is your objective. You need to extract the key information or the assumptions to be made and transform into a mathematical model to represent the problem. The mathematical problem is solved and you reflect on any solutions to decide if the model has been adequate to solve the original problem or revise the assumptions.

We will complete this introduction to mathematical modelling with one more model that can be used to attempt to solve the problem of a fair voting system.

The D'Hondt Method is extensively used in European voting systems for allocating seats with proportional representation. Seats in the European Parliament representing member countries, including England, Scotland and Wales, are distributed according to a model of proportional representation.

As an example, consider a city election when there are 5 seats to allocate and the votes cast are as shown below. It is easy to show how 3 of the 5 seats should be allocated but who deserves the other 2?

Party	Votes
A	17 920
B	11 490
C	11 170
D	4 420

There are a number of models used in practice to allocate seats in a proportional representation system. To use the D'Hondt Method, divide each of the total votes cast for each party by 1, 2, 3, ..., as shown in the next table and then allocate the five seats to the five largest numbers in the table. This gives the numbers starred in the next table, with an allocation **A : 3 B : 1 C : 1 D : 0**.

1.3

*	A	B	C	D
1	17 920 *	11 490 *	11 170 *	4420
2	8960 *	5745	5585	2210
3	5973 *	3830	3723	1473
4	4480	2873	2793	1105

This may look a little harsh on Party D but there is no perfect solution. There is a Scandinavian compromise method which uses the divisors 1.4, 3, 5, 7, ..., which gives a different allocation not a perfect solution!

Mathematical modelling is an important process that underpins the way we apply mathematics and will be an important component of Core Maths courses. It is used in a variety of contexts ranging from satellite launches to population growth and economic forecasting.

It is important to note though that whilst a mathematical model gives an exact answer, the interpretation of this as a solution to a real world problem can be less than perfect. This is the challenge of using and applying mathematics!

Bar code design

Left hand codes

