

Further Calculus

Essential information

- Second derivatives**

If y is a function of x , then $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

e.g. If $y = x^3$, $\frac{dy}{dx} = 3x^2$, $\frac{d^2y}{dx^2} = 6x$

- Stationary points**

If y is a function of x , then stationary points are defined by values of x for which $\frac{dy}{dx} = 0$.

e.g. If $y = x^2 + x - 6$, $\frac{dy}{dx} = 2x + 1$
 $= 0$ when $x = -\frac{1}{2}$

so y has stationary point at $x = -\frac{1}{2}$.

- Maximum and minimum points**

If y is a function of x and $\frac{dy}{dx} = 0$ at $x = a$, then it is a

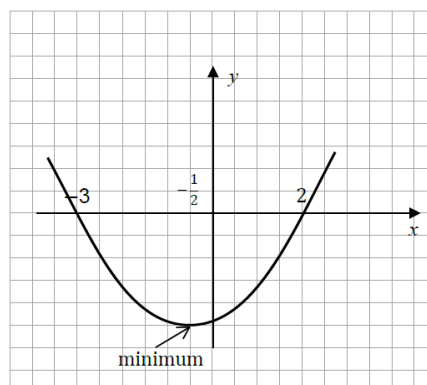
(i) maximum value of y if $\frac{d^2y}{dx^2} < 0$ at $x = a$

(ii) minimum value of y if $\frac{d^2y}{dx^2} > 0$ at $x = a$

e.g. If $y = x^2 + x - 6$, $\frac{dy}{dx} = 2x + 1 = 0$ at $x = -\frac{1}{2}$

$$\frac{d^2y}{dx^2} = 2 > 0 \text{ for any value of } x.$$

Hence minimum at $x = -\frac{1}{2}$ (see sketch of function)



- Point of inflection**

If y is a function of x such that $\frac{dy}{dx} = \frac{d^2y}{dx^2} = 0$ at $x = a$ but $\frac{d^3y}{dx^3} \neq 0$ at $x = a$, then $x = a$ is a point of inflection.

e.g. If $y = x^3 - 3x^2 + 3x - 1$

$$\frac{dy}{dx} = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2$$

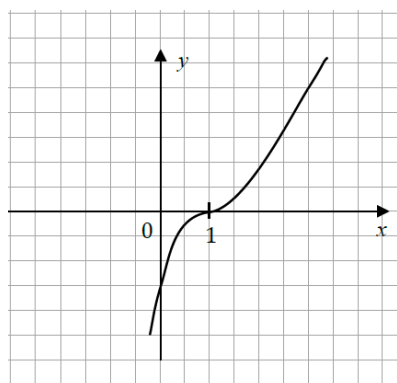
$$= 0 \text{ at } x = 1$$

$$\frac{d^2y}{dx^2} = 6x - 6$$

$$= 0 \text{ at } x = 1$$

$$\frac{d^3y}{dx^3} = 6x \neq 0 \text{ at } x = 1$$

Hence $x = 1$ is a point of inflection (see sketch below).



- Composite functions**

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

- Note also that $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

e.g. If $y = 4x + 1$, $\frac{dy}{dx} = 4$

Also $x = \frac{1}{4}(y - 1)$ so $\frac{dy}{dx} = \frac{1}{\frac{1}{4}}$, confirming the results above.

- Exponential functions**

$$\frac{d}{dx}(e^x) = e^x \text{ and}$$

$$\frac{d}{dx}(e^{kx}) = ke^{kx} \text{ for constant } k$$

- **Integration: key results**

$$\int (ax + b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + k \quad (n \neq -1)$$

$$\int \frac{1}{(ax+b)} dx = \frac{1}{a} \ln(ax + b) + k$$

$$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + k$$

- **Differentiating products and quotients**

If $y(x) = u(x) v(x)$, then

$$\frac{dy}{dx} = \frac{du}{dx} v + u \frac{dv}{dx}$$

e.g. $y = x^2 \sin x$, then $u = x^2$, $v = \sin x$ and

$$\frac{dy}{dx} = 2x \sin x + x^2 \cos x$$

If $y(x) = \frac{u(v)}{v(x)}$, then

$$\frac{dy}{dx} = \frac{\frac{du}{dx} v - u \frac{dv}{dx}}{v^2}$$

e.g. $y = \frac{x^2}{1-x}$, $u = x^2$, $v = 1-x$

$$\frac{dy}{dx} = \frac{2x(1-x) - x^2(-1)}{(1-x)^2}$$

$$= \frac{-x^2 + 2x}{(1-x)^2}$$