

ONE-DIMENSIONAL MOTION

Text

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One-Dimensional Motion

Introduction

This unit will introduce a more mathematical approach to solving problems in mechanics. The *Mechanics Introduction* Unit discussed motion and how forces affect motion (known as **Dynamics**); this will be covered in much more detail here, in Sections 4 and 5.

Sections 1 to 3 will not be concerned with forces, but will describe the motion of objects, in terms of position (displacement), velocity and acceleration. This study of motion, without considering the forces that produce the motion, is known as **Kinematics**.

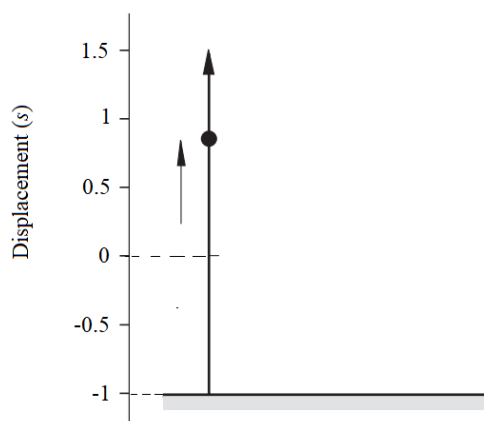
This unit will concentrate on the description of objects which move along a **straight line**. This is one-dimensional motion, a good starting point, from there the concepts can later be extended to two dimensions, then to the three spatial dimensions of the world around you.

1 Representing Motion

Displacement

Displacement is the measure of the position of an object. If you throw a ball upwards, you can measure its position by its **displacement**, as shown on the right. The displacement can be set as 0 for the ball leaving your hand, then the height above this is measured, in metres, as s .

Note that the displacement can take negative values, such as -1 for when the ball falls back down to the level of the ground.



The ball is modelled as though it is *moving along a straight line* here, travelling directly upwards along a straight line in the air, then falling back through the same line, passing through the point it leaves your hand and continuing downwards.



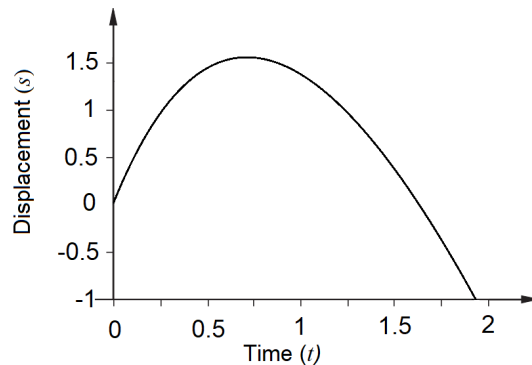
Note

Objects in this unit will be treated as **particles**. This was discussed in the last unit and means that we ignore the size, shape and spin of the object, so to model an object like a tennis ball, a point within the ball is used, usually at the centre of it.

Throughout the study of mechanics you will find that assumptions similar to the particle assumption are used in order to simplify problems and to allow them to be solved. These will be introduced as we progress through the units.

Displacement - Time graphs

As a particle moves, the displacement changes so that s is a function of time, t . A graphical method of showing motion is with a **displacement - time graph** which is a plot of s against t . In the example of the ball being thrown upwards and falling back to the ground, the graph would look like the one on the right.



This type of graph is very useful to help visualise the motion of the ball. Key positions and displacements, can be identified on the graph:

- the time taken to reach maximum height is at about 0.75 seconds
- the time taken to rise and fall back down and hit the ground is a little under 2 seconds
- the maximum height reached above the hand of the person throwing the ball is about 1.5 metres
- the height on hitting the ground is -1 metres, so the ball must have left the thrower's hand at 1 metre above the ground; the maximum height it reached above the ground, therefore, was about 2.5 metres.



Note

There is an important distinction between the terms **displacement** and **distance**.

Displacement has direction: In one-dimensional space this means that it can be positive or negative. The final displacement of the ball is -1 metres.

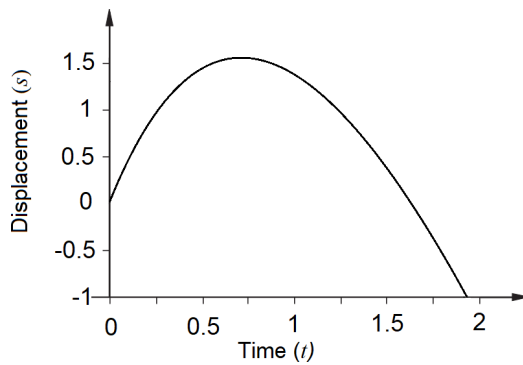
Distance has no direction, the minus sign is removed. So the total distance travelled by the ball can be found by adding up the distance travelled to the maximum height and the distance to fall back down to the ground: $1.5 + 2.5 = 4$ metres. To find *distance* the *magnitude of the displacements* have been found, meaning a displacement of -2.5 metres becomes 2.5 metres: minus sign removed.

Be aware that the graph represents the movement of an object *along a line*, in one dimension, the second dimension on the graph is for time, not space.

Velocity

The gradient of the Displacement-time graph represents velocity. A positive gradient indicates that the ball is travelling upwards, negative gradient for falling down. The steeper the gradient, the greater the magnitude of the velocity of the ball. Key observations for velocity are:

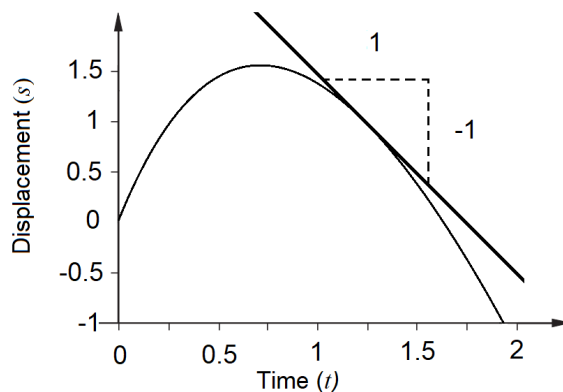
- the graph starts off steep, then become less steep before reaching the maximum height, indicating that the ball is slowing down as it travels upwards, as you would expect
- after reaching maximum height the steepness of the graph back down increases as time advances, confirming that the ball is speeding up as it falls to the ground
- at the maximum height the graph levels off, indicating that the gradient is zero, at this point the ball momentarily stops, before falling back down to the ground



Note

There is an important distinction between terms **velocity** and **speed**. Velocity has direction, in one dimensional motion it can be positive or negative; speed is the magnitude of the velocity and is always positive. On the graph a little after 1 second the velocity is -1 ms^{-1} because at this point for the gradient:

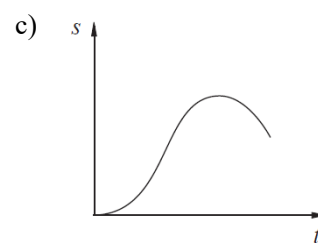
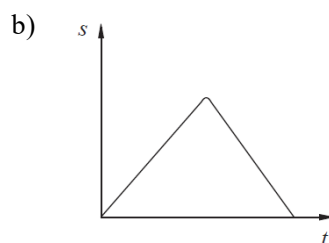
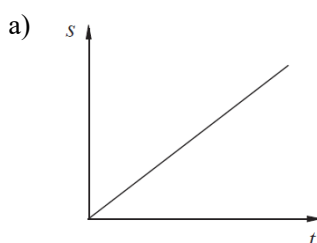
$$\text{gradient} = \frac{-1}{1} = -1$$



The velocity has a negative sign because the ball is travelling down, towards the ground. The speed, however, would be 1 ms^{-1} . Speed has magnitude only (no sign to indicate direction); for speed it doesn't matter which direction the ball is travelling in, the ball is travelling 1 metre in 1 second.

Worked Example 1

Describe the motion represented by a particle represented in each of the displacement-time graphs shown here. Remember that we are dealing in this section with an object moving along a line.





Solutions

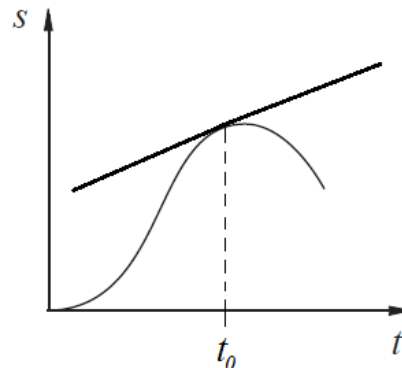
- a) The particle moves along a line at a constant velocity.
- b) The particle moves along a line at a constant positive velocity, then immediately reverses direction and begins to move in the opposite direction, with a constant negative velocity.
- c) The particle slowly speeds up and then slows down, it reverses direction and begins to return to its starting point.

The average velocity of a particle over a given time period will be familiar to you and is defined by

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{change in time}}$$

For the previous examples the first two cases have straight lines, so the above equation will give you the actual velocity for the particle at all times along the straight line section of the graph, also equal to the average velocity.

For a curved section you can use the tangent of the curve to find the velocity at a point in time t_0 as shown here. In calculus terms this is equivalent to finding the derivative of the function of s with respect to time t and substituting in the value at t_0 . So if $s(t)$ is the function for the displacement s in terms of time t , finding the velocity at t_0 is achieved by differentiating s with respect to t , then substituting in the value t_0 .



Thus velocity is defined in the following way.

If $s(t)$ is the displacement of a particle, then its velocity is defined by

$$v = \frac{ds}{dt}$$

The velocity will be a function of time, $v(t)$, so can be used to find the velocity at any time.



Note

Function notation is used here: the t in brackets in $s(t)$ emphasises that s is a function of time t . This notation will be used frequently but do not be confused, s and $s(t)$ both just represent displacement. Also throughout these sections v or $v(t)$ may be used to represent velocity and a or $a(t)$ may be used to represent acceleration.



Worked Example 2

The displacement of a particle is given by

$$s = t^2 + 2t$$

With s in metres and time, t , in seconds. Calculate:

- the displacement of the particle when $t = 3$
- the velocity v of the particle when $t = 3$



Solution

- The displacement s is given by is given by

$$s = t^2 + 2t$$

with $t = 3$ we calculate:

$$s = 3^2 + 2 \times 3$$

$$= 15 \text{ m}$$

- The velocity v is found by differentiating s :

$$v = \frac{ds}{dt} = 2t + 2$$

with $t = 3$ we calculate:

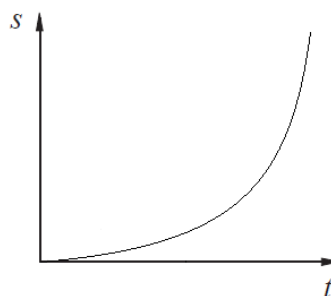
$$v = 2 \times 3 + 2$$

$$= 8 \text{ ms}^{-1}$$

So it can be seen from this example that given $s(t)$, much information can be found about the velocity of the particle. Starting at a time t of zero, the graph of s against t for the previous example, $s = t^2 + 2t$, looks like this:

From the graph it can be seen that the displacement increases as time increases. It can also be seen that the gradient of the curve increases as time increases. This confirms the first derivative,

$$\frac{ds}{dt} = 2t + 2$$



which does increase as t increases. So this particle, as it moves along a line, will continue to move further away from its starting point and will speed up the as time goes on.



Worked Example 3

The displacement of a particle is given by

$$s(t) = t(3 - t)$$

With s in metres and time, t , in seconds. Calculate:

- the displacement of the particle when $t = 2$
- the time when the particle is at rest (this is when velocity v of the particle is zero)



Solution

- The displacement s is given by

$$s(t) = t(3 - t)$$

with $t = 2$ we calculate:

$$s(2) = 2 \times (3 - 2)$$

$$= 2 \text{ m}$$

- The velocity v is found by differentiating s :

$$\frac{ds}{dt} = 3 - 2t$$

(this was found by multiplying out the bracket to give $s = 3t - t^2$, then differentiating)

we then set $v = 0$ or, equivalently, the left hand side $\frac{ds}{dt} = 0$:

$$0 = 3 - 2t$$

which solves to give

$$t = 1.5 \text{ seconds}$$

Having the function of displacement in terms of time allows you to differentiate and find velocity at any time. Expanding on this, if you are given the function $v(t)$, for velocity in terms of time, calculus lets you find the displacement of the particle at any time by integrating, using integration as the reverse of differentiation. This will be explored later.

But continuing for now with displacement as our starting point, we can go further than differentiating once to obtain velocity: we can differentiate again to obtain acceleration.

Acceleration

Acceleration is the change in velocity with time. On a long journey by car or train, the journey begins with acceleration to increase the velocity of the vehicle. Commonly you begin with a velocity of zero and accelerate to increase your velocity to 70 miles per hour.

You have seen that velocity is defined as the rate of change of displacement with respect to time. The acceleration can be defined in a similar way, namely as the rate of change of velocity with respect to time. We can, therefore, define acceleration as follows.

If $v(t)$ is the velocity of a particle at time t , then the acceleration of the particle is defined by

$$a = \frac{dv}{dt}$$

It follows from this that given the displacement, it will be possible to find the acceleration by differentiating once to get velocity, then differentiating again to obtain acceleration.



Worked Example 4

The displacement of a particle is given by

$$s = 5t^3 - 3t^2$$

With s in metres and time, t , in seconds.

- Find the acceleration as a function of time.
- Find the acceleration of the particle when $t = 2$



Solution

- First the velocity v is found by differentiating s :

$$\frac{ds}{dt} = 15t^2 - 6t$$

so

$$v = 15t^2 - 6t$$

Then the velocity can be differentiated to find the acceleration:

$$\frac{dv}{dt} = 30t - 6$$

So

$$a = 30t - 6$$

b) When $t = 2$

$$a = 30 \times 2 - 6$$

$$a = 54 \text{ ms}^{-2}$$

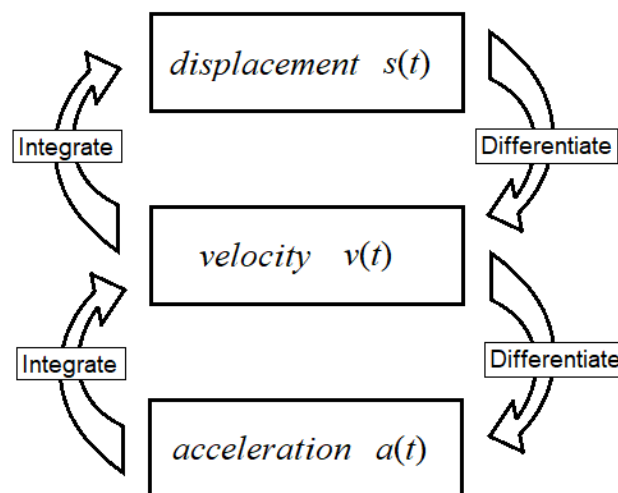
Relationships between displacement, velocity and acceleration

In this section you have seen that the motion of a particle along a straight line can be described by displacement $s(t)$. The velocity and acceleration of the particle are then given by

$$v = \frac{ds}{dt} \quad \text{and} \quad a = \frac{dv}{dt}$$

We know that the reverse of differentiation is integration. It follows from this that displacement can be found by integrating velocity and velocity can be found by integrating acceleration.

This is summarised in the following diagram, where integration and differentiation are taken with respect to time.



In mechanics problems, the acceleration is often known and the velocity and displacement have to be found. Note that when you integrate you will introduce an unknown constant, a '+ c' which you can find if you are given further information, as will be seen in the following examples.



Worked Example 5

The acceleration of a particle is given as a function of time, t :

$$a = 2 + t.$$

Given also that the particle starts at rest at a displacement of $s = 0$, find the displacement of the particle in terms of time.



Solution

Using the relationships discussed, we know that we can find the velocity by integrating the acceleration:

$$v = \int a \, dt = \int (2 + t) \, dt$$

So

$$v = 2t + \frac{t^2}{2} + c_1$$

where c_1 is a constant.

But the constant c_1 can be found with the additional information that the particle starts at rest, which is interpreted that at $t = 0, v = 0$, we can substitute these values into the above equation to find c_1 :

$$0 = 2 \times 0 + \frac{0^2}{2} + c_1$$

Giving $c_1 = 0$

So

$$v = 2t + \frac{t^2}{2}$$

Now we can go on to find the displacement by integrating the velocity:

$$s = \int v \, dt = \int 2t + \frac{t^2}{2} \, dt$$

So

$$v = t^2 + \frac{t^3}{6} + c_2$$

where c_2 is a constant.

The constant c_2 can be found with the additional information that the particle starts at $s = 0$, so at $t = 0, s = 0$, we can substitute these values into the above equation to find c_2 :

$$0 = 0^2 + \frac{0^3}{6} + c_2$$

giving $c_2 = 0$.

So

$$s = t^2 + \frac{t^3}{6}$$



Worked Example 6

The velocity of a particle is given as a function of time, t :

$$v = 3t + 2.$$

Given also that the particle starts at a displacement of $s = 10$, find

- the displacement of the particle in terms of time.
- the acceleration of the particle.



Solution

- Using the relationships discussed, we know that we can find the velocity by integrating the acceleration:

$$s = \int v \, dt = \int (3t + 2) \, dt$$

So

$$s = \frac{3}{2}t^2 + 2t + c$$

where c is a constant.

But the constant c can be found with the additional information that the particle starts at a displacement of $s = 10$. So $s = 10$ when $t = 0$. We can substitute these values into the above equation to find c :

$$10 = \frac{3}{2}0^2 + 2 \times 0 + c$$

Giving $c = 10$

Finally,

$$s = \frac{3}{2}t^2 + 2t + 10$$

- We can find the acceleration by differentiating the velocity:

$$v = 3t + 2$$

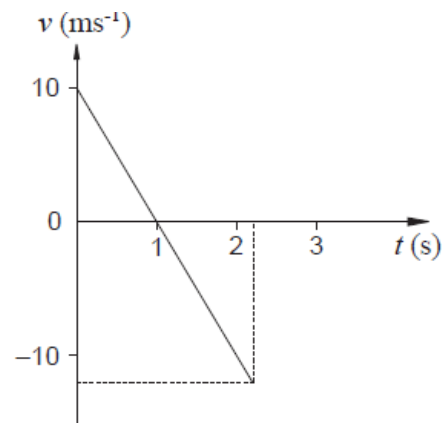
$$a = \frac{dv}{dt} = 3$$

So the particle has an acceleration of 3 ms^{-2} , a constant acceleration.

Velocity-time graphs

The graph of the velocity against time is called a **velocity-time graph**. The figure shows the velocity-time graph for the motion of a ball thrown up into the air then falling to the floor. Positive velocity is taken to indicate upwards motion.

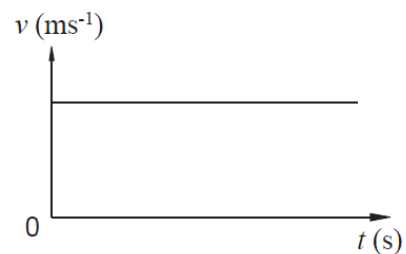
The ball begins its motion with a velocity of 10 ms^{-1} and this velocity falls to zero during the first second of the motion as the ball moves upwards, while slowing down. It reaches its maximum height, with a velocity of zero momentarily. The velocity then continues to decrease, the ball travelling at approximately -12 ms^{-1} when it hits the floor.



Note how velocity has a sign so contains information about direction. A minus sign indicates that the ball is travelling in a downwards direction. The speed of the ball when it hits the ground is about 12 ms^{-1} .

A further example is shown in the diagram on the right the velocity - time graph for an object moving at a constant velocity.

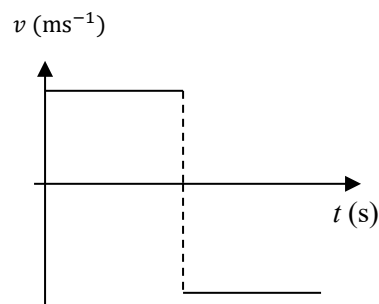
This could be the motion of an object moving in deep space, with no forces acting upon it to change its motion.



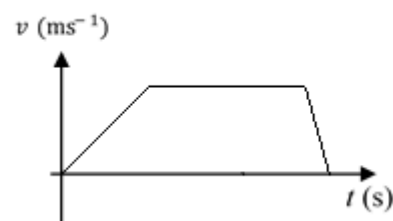
Worked Example 7

Describe the motion of the particles represented in the following graphs.

a)



b)





Solutions

- Constant velocity in one direction, followed by instantaneous change to constant velocity in the reverse direction.
- Increasing velocity, then velocity remaining at a constant level for some period, before finally decreasing.

From the previous section we also know that acceleration is the change of velocity with time. This can be related to the velocity-time graph as the gradient of the line; so the gradient of the line represents acceleration.

If we integrate velocity we obtain displacement. Integrating gives you the area under a graph, so relating this to the velocity - time graph tells us that the area under a velocity-time graph gives you the displacement.

Given a Velocity-time Graph:

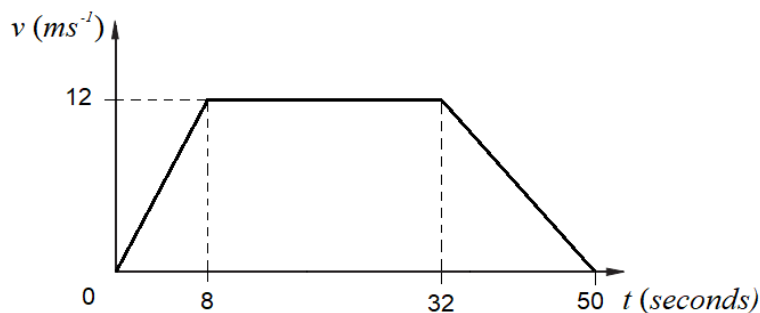
Acceleration can be found by calculating the **Gradient of the Graph**

Displacement can be found by calculating the **Area under the Graph**



Worked Example 8

The following velocity-time graph represents the motion of an object.



- Describe the motion of the object.
- Find the value of the acceleration of the object at the three stages of motion along the path.
- Find the displacement of the object after 50 seconds.



Solutions

- a) Velocity increases from zero at a constant rate for 8 seconds, then remains constant at 12 ms^{-1} for 24 seconds. The velocity then reduces at a constant rate for 18 seconds, ending at zero.
- b) Acceleration is found by calculating the gradient of the velocity-time graph. This breaks down to three sections:

From 0 to 8 seconds:

$$a = \frac{12 - 0}{8 - 0} = \frac{12}{8} = 1.5 \text{ ms}^{-2}$$

From 8 to 32 seconds:

$$a = \frac{12 - 12}{32 - 8} = \frac{0}{24} = 0 \text{ ms}^{-2}$$

Note that you don't necessarily have to do this calculation; the velocity is constant, so the acceleration must be zero, you can see this from the graph

From 32 to 50 seconds:

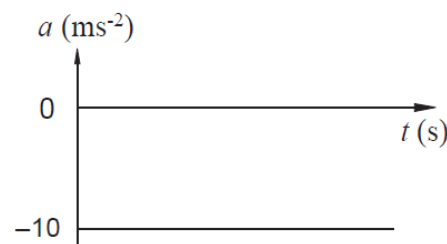
$$a = \frac{0 - 12}{50 - 32} = \frac{-12}{18} = -0.67 \text{ ms}^{-2}$$

- c) Displacement is found by calculating the area under the velocity-time graph. There are many ways to do this, e.g. you can add the areas of the triangle, rectangle and triangle that can be seen on the graph, but here the calculation will be done using the area of a trapezium:

$$s = \frac{1}{2}h(a + b) = \frac{1}{2} \times 12 \times (24 + 50) = 444 \text{ metres}$$

Acceleration-time graphs

You can also create acceleration-time graphs, as shown on the right. Here the acceleration is constant at -10 ms^{-2} . The negative sign indicates that the velocity decreases continuously with time. Taking up to be positive, this could represent the motion for an object being subject to gravity on the Earth's surface, constantly having its velocity reduced back towards the ground.

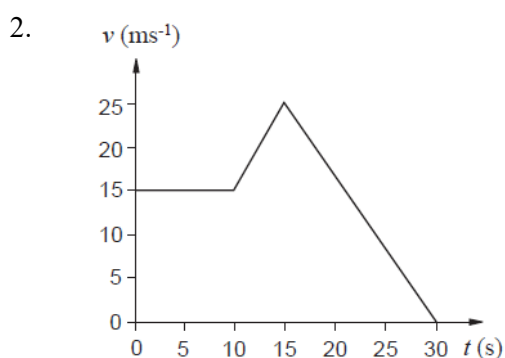




Exercises

1. Sketch displacement-time and velocity-time graphs for the following:

- A car starts from rest and increases its velocity steadily to 10 ms^{-1} in 5 seconds.
The car holds this velocity for another 10 seconds and then slows steadily to rest in a further 10 seconds.
- A ball is dropped on to a horizontal floor from a height of 3 m. The ball bounces several times before coming to rest.
- A person jumps out of an aircraft and falls until the parachute opens. The person glides steadily to the ground.



The diagram represents the motion of an object for 30 seconds. Calculate the acceleration for each of the following intervals:

- $0 < t < 10$
- $10 < t < 15$
- $15 < t < 30$

Calculate the displacement of the object over the 30 seconds.

3.

$t(\text{s})$	0	1	2	3	4	5	6	7	8
$v(\text{ms}^{-1})$	0	8	16	24	32	30	24	14	0

- Plot these figures on a velocity- time graph.
- Verify that for $0 \leq t \leq 4$, the figures are consistent with constant acceleration.
- Calculate an estimate for the acceleration at $t = 6$.
- Calculate an estimate for the displacement after 8 seconds.

4. During the launch of a rocket the velocity was noted every second for 10 seconds and the following table of values obtained.

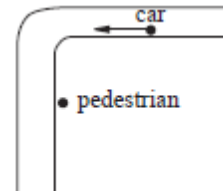
$t(\text{s})$	0	1	2	3	4	5	6	7	8	9	10
$v(\text{ms}^{-1})$	0	32	60	128	176	224	272	320	368	400	448

Estimate the distance travelled by the rocket during the first 10 seconds of its motion.

Estimate the distance travelled by the rocket during the first 10 seconds of its motion.

5. A road has a sharp bend around which approaching traffic cannot be seen. The width of the road is 6 m and the speed limit is 30 mph.

The bend is on the near side of the road. How far from the bend should a pedestrian cross the road to safely reach the other side before a car arrives? Assume the pedestrian walks at 3 mph.



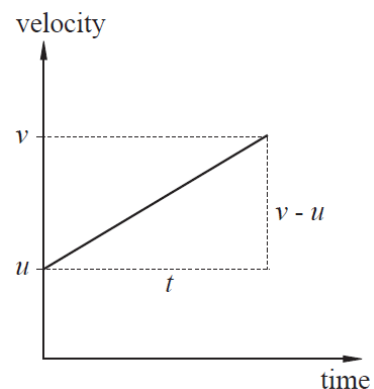
6. A particle is set in motion at time $t = 0$ and its position is subsequently given by $s = 4 + 5t - 2t^2$.
- Calculate the velocity of the particle after 1 second and after 2 seconds. What is the speed and direction of motion at each of these times?
 - Find the time at which the particle is instantaneously at rest.
 - Calculate the acceleration of the particle at $t = 1$ s and $t = 2$ s.
 - Describe the motion of the particle.
7. Repeat Problem 6 for
- $s = t^2 - 4t + 1$
 - $s = 3 + 18t - 7.5t^2 + t^3$
8. The acceleration of a particle is given by $a = -10 \text{ ms}^{-2}$. At the instant $t = 1$ s, the particle is at position $s = 2$ m and has velocity 3 ms^{-1} .
- Find the velocity and displacement of the particle as functions of time.
 - Calculate the position and velocity of the particle when $t = 2$ s.

2 Modelling Motion under Constant Acceleration

There are several simple formulae which can be used when dealing with motion under **constant** (also known as uniform) **acceleration**. Some of these can be derived by using the relationships established in the last section.

The velocity-time graph on the right shows the motion of an object with initial velocity u and final velocity v after t seconds have elapsed. The gradient of the line is calculated from the expression

$$\frac{v - u}{t}$$



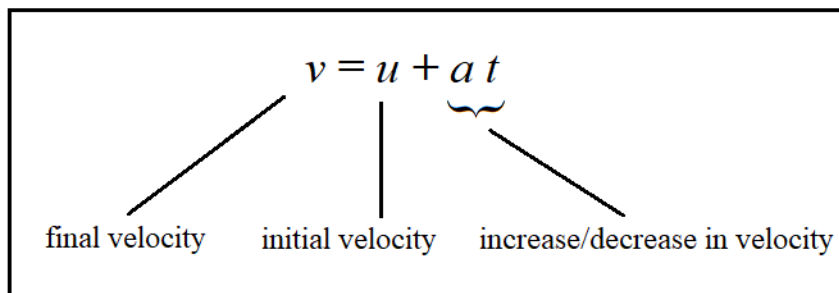
Since the gradient is equal to the value of the acceleration, a , then an equation for the acceleration can be derived:

$$a = \frac{v - u}{t}$$

This can be rewritten as

$$v = u + at \quad (1)$$

The equation can be understood in the following way:

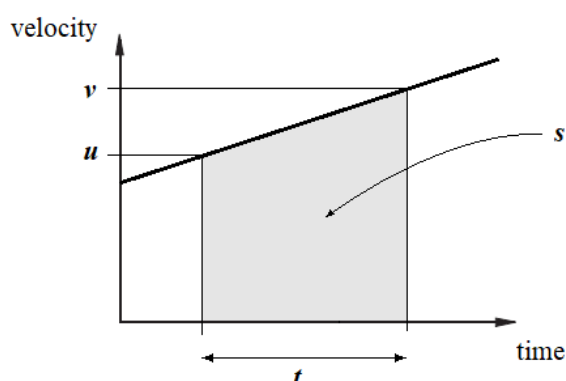


So acceleration can be positive, resulting in an increase in velocity, or negative, resulting in a decrease in velocity.

We can derive more of these equations for an object travelling with constant acceleration using the following variables:

u	initial velocity
v	final velocity
a	acceleration
s	distance
t	time

Another equation can be derived from using the fact that the area under the velocity - time graph is equal to the displacement, s , of the object:



The area under the graph forms a trapezium, then the formula for the area of a trapezium produces the following relationship:

$$s = \frac{(u + v)}{2} t$$

This equation can be understood in the following way:

The diagram shows the equation $s = \frac{(u + v)}{2} t$ inside a rectangular box. Three lines point from labels below to parts of the equation: 'displacement' points to s , 'average velocity' points to $\frac{(u + v)}{2}$ (with a curly brace under the fraction), and 'time' points to t .

The equation can then be related to the familiar relationship of *distance* = *speed* $\times$ *time*.

Note that the two equations established so far each contain four variables. Three more equations such as these can be derived, giving a total of five equations relating s , u , v , a and t for an object moving with a constant acceleration:

A rectangular box containing the following five equations of motion:

$$v = u + at$$

$$s = \frac{(u + v)}{2} t$$

$$s = ut + \frac{1}{2} at^2$$

$$v^2 = u^2 + 2as$$

$$s = vt - \frac{1}{2} at^2$$

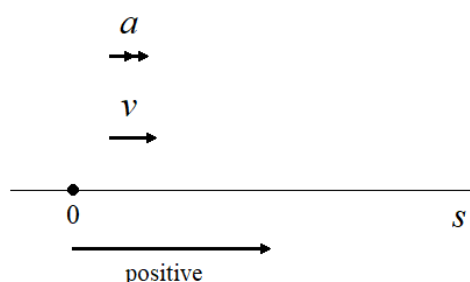
The three new equations are not as intuitive as the first two and can be derived algebraically by combining the first two, e.g. substituting v from the first equation into the second equation, then simplifying, will give you the third equation.

These equations can be applied to problems involving objects moving with constant acceleration; they are commonly referred to as the *suvat* equations.



Note

When solving problems it is important to establish the positive direction. Here the positive direction is from left to right. Commonly the displacement s will be in reference to a starting point 0. Then throughout the problem a and v must hold to the same sign convention.



Left to right is a common framework. Upwards as the positive direction is also common, for an object travelling up but being acted on by gravity, with its negative value.



Worked Example 1

A motorbike accelerates at a constant rate of 3 ms^{-2} . Calculate

- the time taken to accelerate from 20 mph to 40 mph,
- the distance in metres covered during this time.



Solution

You can use equation (1) to find the time and then equation (2) to find the distance travelled, but first you must convert mph to ms^{-1} .

Using the conversion rule $5 \text{ miles} \approx 8 \text{ kilometres}$ gives

$$\begin{aligned} 10 \text{ mph} &= 16 \text{ kph} = \frac{16 \times 1000}{60 \times 60} \text{ ms}^{-1} \\ &= 4.445 \text{ ms}^{-1} \end{aligned}$$

Hence $20 \text{ mph} = 8.89 \text{ ms}^{-1}$ and $40 \text{ mph} = 17.78 \text{ ms}^{-1}$.

- Using $v = u + at$, the time taken to accelerate from 8.89 ms^{-1} to 17.78 ms^{-1} at 3 ms^{-2} is given by

$$t = \frac{17.78 - 8.89}{3} = 2.96 \text{ s (to 3 sig.fig.)}$$

- Using $s = \frac{(u+v)}{2}t$, the distance travelled in this time is

$$s = \left\{ \frac{8.89 + 17.78}{2} \right\} \times 2.96 = 39.5 \text{ m}$$



Worked Example 2

A car accelerates from a velocity of 16 ms^{-1} to a velocity of 40 ms^{-1} in a distance of 500 m.

Calculate the acceleration of the car.



Solution

Using

$$v^2 = u^2 + 2as$$

$$40^2 = 16^2 + 2 \times a \times 500$$

$$a = \frac{1600 - 256}{1000} = 1.344 \text{ ms}^{-2}$$



Worked Example 3

A car decelerates from a velocity of 36 ms^{-1} . The magnitude of the deceleration is 3 ms^{-2} .

Calculate the time required to cover a distance of 162 m.



Solution

When an object is slowing down it is said to be **decelerating**. You then use the *suvat* equations but with a negative value for a .

In this problem set, $a = -3$.

Let t seconds be the time required to cover 162 m.

Using

$$s = ut + \frac{1}{2}at^2$$

$$162 = 36t + \frac{1}{2} \times (-3) \times t^2$$

Rearranging this gives

$$1.5t^2 - 36t + 162 = 0$$

Dividing by 1.5 gives

$$t^2 - 24t + 108 = 0$$

Factorizing gives

$$(t - 6)(t - 18) = 0$$

so that

$$t = 6 \text{ or } 18$$



Exercises

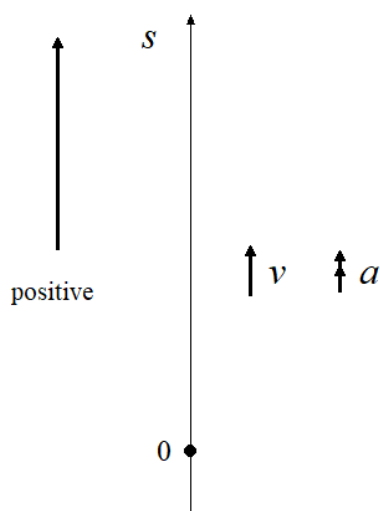
1. A car accelerates uniformly from a speed of 50 kph to a speed of 80 kph in 20 seconds.
Calculate the acceleration in ms^{-2} .
2. For the car in Question 1, calculate the distance travelled during the 20 seconds.
3. A train signal is placed so that a train can decelerate uniformly from a speed of 96 kph to come to rest at the end of a platform. For passenger comfort the deceleration must be no greater than 0.4 ms^{-2} . Calculate
 - (a) the shortest distance the signal can be from the platform;
 - (b) the shortest time for the train to decelerate.
4. A rocket is travelling with a velocity of 80 ms^{-1} . The engines are switched on for 6 seconds and the rocket accelerates uniformly at 40 ms^{-2} .
Calculate the distance travelled over the 6 seconds.

5. In 1987 the world record for the men's 60 m race was 6.41 seconds.
 - (a) Assuming that the race was carried out under constant acceleration, calculate the acceleration of the runner and his speed at the end of the race.
 - (b) Now assume that in a 100 m race the runner accelerates for the first 60 m and completes the race by running the next 40 m at the speed you calculated in (a). Calculate the time for the athlete to complete the race.
6. The world record for the men's 100 m was 9.83 s in 1987. Assume that the last 40 m was run at constant speed and that the acceleration during the first 60 m was constant.
 - (a) Calculate this speed.
 - (b) Calculate the acceleration of the athlete.
7. Telegraph poles, 40 m apart stand alongside a railway line. The times taken for a locomotive to pass the two gaps between three consecutive poles are 2.5 seconds and 2.3 seconds respectively. Calculate the acceleration of the train and the speed past the first post.
8. The world record for the women's 60 m and 100 m are respectively 7.00 seconds and 10.49 seconds. Analyse this information using the method given in Question 6.
9. A set of traffic lights covers road repairs on one side of a road in a 30 mph speed limit area. The traffic lights are 80 m apart so time must be allowed to delay the light changing from green to red. Assuming that a car accelerates at 2 ms^{-2} what is the least this time delay should be?
10. A van travelling at 40 mph skids to a halt in a distance of 15 m. Find the acceleration of the van and the time taken to stop, assuming that the deceleration is uniform.

3 Acceleration and Gravity

When solving problems on the Earth's surface, the effect of gravity can be treated as a constant downwards acceleration on an object. The value $a = 9.81 \text{ ms}^{-2}$ is fairly accurate and has been found by theory and experimentation, but $a = 10 \text{ ms}^{-2}$ is commonly used for convenience, it is a fairly good estimate and this is the value that will be used in the following problems.

Note that in reality the effect of gravity varies depending on where you are on the Earth's surface and how far you are from the centre of the Earth, as discussed in the first unit. If you are further away from the Earth's surface, say in a space station, the acceleration due to gravity will be smaller; letting $a = 10 \text{ ms}^{-1}$ is another example of an assumption used to help in problem solving in mechanics.



The sign convention for problems involving gravity can be as shown on the left, with positive being the 'up' direction. Using this convention, s , v and a will all have positive signs when directed upwards and a negative sign when directed downwards.

For acceleration, a , a positive sign indicates acceleration upwards. A negative sign indicates deceleration, or a 'pull' downwards. Importantly this applies to gravity, so in this convention the value $a = -10 \text{ ms}^{-2}$ would be used.

For s , a position of zero is set (often ground level) as the starting position, then positive s is above this position (height) and negative s is below it.

For v , a positive sign indicates that the direction of travel is upwards, a negative sign indicates that the direction of travel is downwards.



Worked Example 1

A ball is thrown vertically upwards with an initial speed of 30 ms^{-1} . Calculate the height reached.



Solution

Since the ball slows down, take $a = -g = -10 \text{ ms}^{-2}$.

Using

$$v^2 = u^2 + 2as$$

$$0 = 30^2 + 2 \times -10 \times h$$

where h is the maximum height reached.

Thus

$$h = \frac{900}{20} = 45 \text{ (to 2 significant figures).}$$



Worked Example 2

A ball is thrown vertically upwards with a speed of 40 ms^{-1} .

Calculate the time interval between the ball being at a height of 20 m on its journey up and the time when it passes back through the 20 m point.



Solution

Using $s = ut + \frac{1}{2}at^2$ with $a = -g = -10 \text{ ms}^{-2}$

$$20 = 40t - 0.5 \times 10 \times t^2$$

where t seconds is the time passed since the ball was thrown up. This rearranges to

$$5t^2 - 40t + 20 = 0$$

$$t^2 - 8t + 4 = 0$$

Using the quadratic formula

$$t = \frac{8 \pm \sqrt{48}}{2} = 7.46 \text{ or } 0.54$$

The ball is 20 m above the point of release twice, at $t = 0.54 \text{ s}$ (on way up) and at $t = 7.46 \text{ s}$ (on way down). The required time interval is $7.46 - 0.54 = 6.92$ seconds.



Exercises

(Take $g = 10 \text{ ms}^{-2}$)

1. A ball is dropped on to level ground from a height of 20 m.
 - (a) Calculate the time taken to reach the ground.
The ball rebounds with half the speed it strikes the ground.
 - (b) Calculate the time taken to reach the ground a second time.
2. A stone is thrown down from a high building with an initial velocity of 4 ms^{-1} . Calculate the time required for the stone to drop 30 m and its velocity at this time.

3

3. A ball is thrown vertically upwards from the top of a cliff which is 50 m high. The initial velocity of the ball is 25 ms^{-1} . Calculate the time taken to reach the bottom of the cliff and the velocity of the ball at that instant.
4. The diagram show three positions of a ball which has been thrown upwards with a velocity of $u \text{ ms}^{-1}$.

Position A is the initial position.



Position B is halfway up.



Position C is at the top of the motion.



Copy the diagram and for each position put on arrows where appropriate to show the direction of the velocity.

On the same diagram, put on arrows to show the direction of the acceleration.

5. If the Earth is assumed to be a perfect sphere then the acceleration due to gravity at a height h m above the surface of the Earth is given by

$$\frac{k}{(R+h)^2}$$

where R is the radius of the Earth in metres and k is a constant.

- Given that $R = 6400 \text{ km}$ and at the Earth's surface $g = 9.8 \text{ ms}^{-2}$, estimate the value of k .
 - Use your calculator to find the height at which the acceleration differs by 1% from its value at the Earth's surface.
 - Use a graphic calculator to draw the variation of acceleration with height.
6. When a ball hits the ground it rebounds with half of the speed that it had when it hit the ground. If the ball is dropped from a height h , investigate the total distance travelled by the ball.
7. One stone is thrown upwards with a speed of 2 ms^{-1} and another is thrown downwards with a speed of 2 ms^{-1} . Both are thrown at the same time from a window 5 m above ground level.
- Which hits the ground first?
 - Which is travelling fastest when it hits the ground?
 - What is the total distance travelled by each stone?

4 Force and Change in Motion

Newton's First and Second Laws

The Mechanics Introduction Unit discussed how force affects motion. We now return to this idea and introduce a mathematical formulation of this principle using Newton's laws.

As described in the Introduction unit, a force can

- start motion
- stop motion
- change the direction of motion.
- make an object move faster or slower

When there are no forces acting on an object, it will remain at rest or it will continue in a straight line at a constant velocity.

This relationship is known as Newton's First Law of Motion:

Newton's First Law

A body remains in a state of rest or moves with uniform motion, unless acted on by a force.

So if the total force acting on a body is **zero** then the body remains at rest or moves with uniform motion, i.e. at a constant velocity. Newton's First Law defines what happens to the motion of a body if no force is present.

When a force does act on a body, however, there is a change in motion.

The physical law relating change in motion and the force acting on a body is given by Newton's Second Law, as applied to objects with constant mass:

Newton's Second Law

The force on an object is equal to its mass multiplied by acceleration:

$$F = ma$$

where

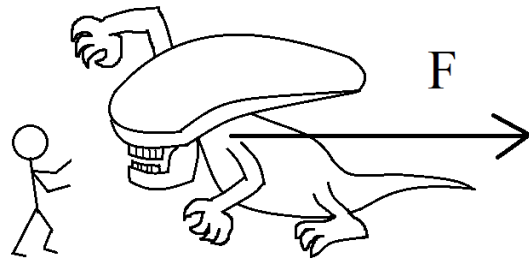
m is in kilograms (kg),

a is in metres per second per second (ms^{-2})

F is in Newtons (N)

This law relates force to acceleration, via the mass of the object. It tells you that to push and accelerate an object with a larger mass, you need to apply a larger force.

For example, imagine that you want to apply a force to a large alien to push it off your spacecraft into the depths of space. If the alien was human-sized, this could be achieved fairly easily with a reasonably sized force; if the alien was a lot larger, perhaps the size of an elephant, you would need a much greater force to push it away from your ship.



One newton is the force required to produce an acceleration of one ms^{-2} in a body of mass one kg. The abbreviation for a newton is N.

Worked Example 1

A car of mass 1100 kg can accelerate from rest to a speed of 30 mph in 12 seconds. Calculate the driving force required to achieve this.

Solution

To find the force we first we need to find the acceleration required to gain the speed of 30 mph. The units must be converted into SI units to apply the $F = ma$ formula. Using the conversion $10 \text{ mph} = 4.47 \text{ ms}^{-1}$ gives $30 \text{ mph} = 13.4 \text{ ms}^{-1}$.

Assuming that the car accelerates at a constant rate, then using $v = u + at$ with $v = 13.4$ along with $u = 0$ and $t = 12$ gives

$$a = \frac{13.4}{12} = 1.12 \text{ ms}^{-2}$$

Applying Newton's Second Law gives

$$\begin{aligned} F &= ma \\ &= 1100 \times 1.12 \\ &= 1230 \text{ N (to 3 significant places).} \end{aligned}$$

Worked Example 2

A force of magnitude 20 N is applied to a particle of mass 4 kg for 6 seconds. Given that the initial velocity of the body is 15 ms^{-1} ,

- calculate the acceleration, a , of the body,
- calculate its velocity, v , after 6 seconds.

After 6 seconds a force, F , is applied to bring the body to rest in a further 125 m.

- Calculate the magnitude of the force.



Solution

- a) Using Newton's Second Law,

$$20 = 4a$$

$$a = 5 \text{ ms}^{-2}$$

- b) Using $v = u + at$, $v = 15 + 5 \times 6 = 45 \text{ ms}^{-1}$

- c) First the acceleration needs to be found, then from this then the force.

Using

$$v^2 = u^2 + 2as$$

$$0^2 = 45^2 + 2 \times a \times 125$$

Solving for a ,

$$a = -\frac{45^2}{250} = -8.1 \text{ ms}^{-2}$$

Notice that the sign is negative, as the acceleration is against the direction of motion.

Using Newton's Second Law,

$$F = 4 \times 8.1 = 32.4 \text{ N}$$

This force is also acting against the direction of motion.

Recall that since all objects fall with an acceleration of g (neglecting air resistance) near the Earth's surface, the force acting on an object of mass m is given by

$$F = mg$$

As described previously, this force is known as the **weight** of the object.



Exercises

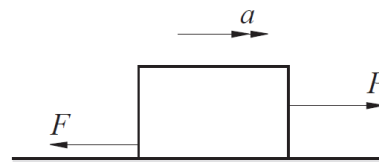
- A force of 50 N is applied to a particle of mass 4 kg for 5 seconds only.
 - Calculate the acceleration for the first 5 seconds.
 - Write down the acceleration for the next 5 seconds.
 - Calculate the distance travelled during the first 10 seconds given that the particle was at rest initially.
- A world class sprinter weighing 80 kg can accelerate from rest to 10 ms^{-1} in about 2 seconds. Estimate the magnitude of the force required to produce this acceleration.
- The brakes of a train are required to bring it to rest from a speed of 80 kph in a distance of 500 m. The mass of the train is 200 tonnes.
 - Calculate the average deceleration.
 - Calculate the average force required to be exerted by the brakes.

4. A toy car of mass 0.04 kg is propelled from rest by an engine which provides a pulling force of $2.7 \times 10^{-3} \text{ N}$ lasting for 8 seconds.
 - (a) Calculate the acceleration.
 - (b) Calculate the velocity after 8 seconds.
 If the speed of the toy car decreases uniformly to zero during the next 32 seconds, find the total distance travelled by the car.
5. A rocket has a mass of 40 tonnes of which 80% is fuel. The rocket motor develops a thrust of 1200 kN at all times.
 - (a) Calculate the acceleration when the rocket is full of fuel.
 - (b) Calculate the acceleration just before the fuel is exhausted.
 - (c) What is the acceleration after the fuel is exhausted ?
6. A parachute reduces the speed of a parachutist of mass 70 kg from 40 ms^{-1} to 10 ms^{-1} in 3 seconds. Calculate the average force exerted by the parachute.
7. A car travelling at 30 mph is brought to rest in 3 seconds during a collision. Calculate the average force exerted on the car during the collision. Assume mass of driver is 70 kg and that of the car is 1100 kg .

5 Multiple Forces

Most problems you solve in mechanics will involve more than one force. To solve these problems you must combine the forces by adding them together, taking into account the direction of the forces via their signs, to establish the *net* force. A good diagram is essential to establish all the forces on the object, and their direction. The types of force you commonly encounter such as friction, tension, etc. were described in the first Unit.

The diagram shows a block of mass $M \text{ kg}$ being pulled by a force $P \text{ N}$ over a horizontal floor. The floor is rough so that there is a force, $F \text{ N}$, due to frictional resistance acting in the opposite direction to P . Left to right is assigned as positive.



Since P and F act in opposite directions, their effect is a net force $P - F$ in the positive direction. Newton's Second Law is then applied using the net force to give

$$P - F = Ma$$

where a is the acceleration of the block

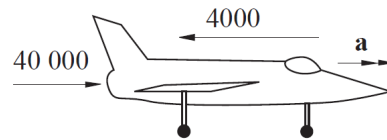
The net force is often called the **resultant** force.

5



Worked Example 1

A jet aircraft of mass 8 tonnes has a single engine which generates a force of 40 000 N. Resistance to motion amounts to a constant value of 4000 N.



[Reminder: 1 tonne = 1000 kg]

- Calculate the acceleration of the aircraft.
- The aircraft starts from rest. Calculate the speed after 12 seconds.

After 12 seconds the engine is switched off.

- Calculate the distance travelled by the aircraft before coming to a halt.



Solution

- The net force acting on the aircraft is $40\,000 - 4000$ N.

So using equation $F = ma$, with net forces on the left hand side

$$40\,000 - 4000 = 8000a.$$

Solving gives $a = 4.5 \text{ ms}^{-2}$.

- Using $v = u + at$ with $u = 0$, $a = 4.5$ and $t = 12$ gives

$$v = 0 + 4.5 \times 12 = 54 \text{ ms}^{-1}$$

- The only horizontal force acting on the aircraft is resistance.

Using $F = ma$

$$0 - 4000 = 8000a$$

which solves to give $a = -0.5 \text{ ms}^{-2}$

where a is the acceleration of the aircraft when the engine is switched off. The negative sign indicates that the aircraft is slowing down.

Using

$$v^2 = u^2 + 2as$$

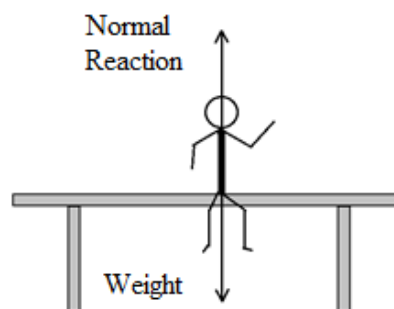
$$0^2 = 54^2 + 2 \times (-0.5) \times s$$

which solves to give $s = 2916$ m where s is the distance travelled by the aircraft.

Newton's Third Law

If you sit on a table, you do not fall to the ground (provided the table is a strong one) but will remain at rest. As established, you have weight, a force acting downwards on you. There must, therefore, be a force equal to your weight which acts upwards and counters your weight so that you do not move.

This force was described in the Introduction Unit and is called the Normal Reaction Force.



This familiar scenario illustrates Newton's third law:

Newton's Third Law

For two bodies in contact, **equal** but **opposite** forces must act from one to the other.

So when you apply a force to an object, the object always applies an equal force back to you. For example, if you are on an ice rink wearing skates (allowing us to assume frictional forces as negligible) and you apply a force to an elephant, also wearing ice skates, you may create in the elephant an acceleration away from you, but there will be an equal force against you, and you will accelerate backwards.

Note that in this scenario, while the forces are equal, your mass is a lot smaller than the elephant's mass, so the application of the law $F = ma$ will tell you that your acceleration will be a lot greater than the acceleration of the elephant. Producing just a small acceleration in the elephant will result in you being accelerated backwards, in the opposite direction to the elephant, at great speed, probably for the best.



Worked Example 2

A manned rocket takes off with an acceleration of 40 ms^{-2} . An astronaut has a mass of 75 kg. Calculate the magnitude of the normal contact force acting from the chair on the astronaut.



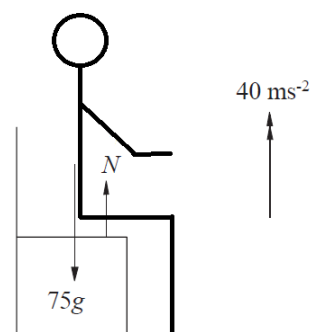
Solution

The normal contact force is N . Using the upwards direction as positive and applying $F = ma$:

$$N - 75g = 75 \times 40$$

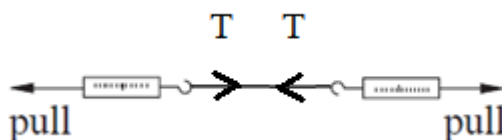
Using $g = 10$, the value of N is 3750 N.

(This means that the astronaut will experience a force of 5 times her/his weight of 750N.)



Connected bodies

Bodies are generally pushed or pulled by strings, bars or chains. In a tug of war match, the opposing teams have no direct contact. If they are not accelerating then the pulling forces must be equal and opposite. Also each member is pulling so there must be an equal and opposite force in the rope. Forces in ropes and strings are known as tensions, as described in the first Unit. Tensions act inwards along the ropes and act in equal but opposite pairs.

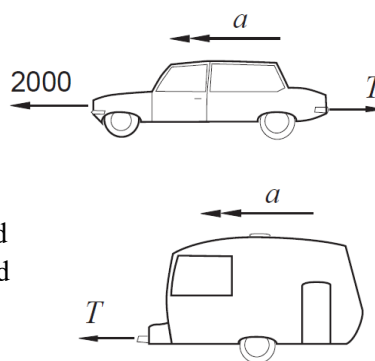


Bars and rods can also support thrusts as well as tensions. Thrusts are denoted by outward facing arrows and they keep objects apart rather than bring them together.

By using equal but opposite forces in connectors, it is possible to solve problems in which more than one body is involved.

Each body can be treated separately by considering the forces acting on it in isolation together with any tensions or thrusts from the connectors.

In the diagram a car of mass 1300 kg is connected to a caravan of mass 700 kg via a coupling. The car engine develops a pulling force of 2 kN. (1 kN = 1000 N)



The motion of the car and caravan can be considered separately. For the caravan, the pulling force is supplied by the tension in the coupling. The car is driven forward by the pulling force of the engine and held back by the tension, T , in the coupling.

For the car, using $F = ma$

$$2000 - T = 1300a$$

where a is the acceleration of the car.

For the caravan

$$T = 700a$$

since the car and caravan must have the same acceleration.

Adding the two equations gives

$$2000 = 2000a$$

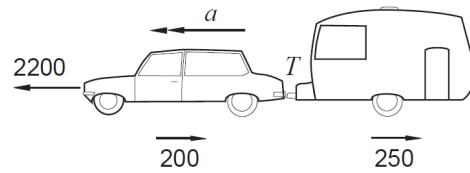
$$\Rightarrow a = 1 \text{ ms}^{-2}$$

The tension in the coupling is $T = 700 \times 1 = 700 \text{ N}$.



Worked Example 3

A car of mass 1100 kg pulls a caravan of mass 800 kg. The force exerted by the engine is 2200 N. In addition, friction and air resistance amount to 200 N on the car and 250 N on the caravan.



Calculate the acceleration of the car and the tension, T , in the coupling between the car and the caravan.



Solution

Using $F = ma$ for the car

$$2200 - 200 - T = 1100a$$

where a is the acceleration of the car.

Using $F = ma$ for the caravan

$$T - 250 = 800a$$

Adding gives

$$1750 = 1900a$$

$$\Rightarrow a = 0.921 \text{ ms}^{-2}$$

Now from

$$T - 250 = 800a$$

$$T = 800 \times 0.921 + 250 = 987 \text{ N}$$

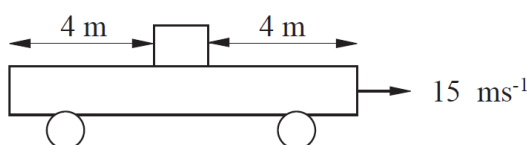
An assumption generally made in the motion of connected bodies is that the mass of the connector can be neglected compared to the masses of the bodies. Such connectors are called 'light'. Another assumption is that the length of the connector remains constant, that is, the connector does not stretch.



Exercises

1. A block of wood of mass 500 gram is pulled by a force of 4 N. The resistance between the wood and the floor is 2 N. Calculate the acceleration.
2. Explain what you feel when you get into a lift which then ascends with an initial acceleration 2 ms^{-2} followed by a steady velocity and finally has a deceleration of 11 ms^{-2} .
3. During a car collision, the car's speed decreases from 50 mph to zero in 4 seconds. The driver is restrained by a car seat belt. Calculate the average force on the chest of an 80 kg driver.

4.



The diagram shows a box on the top of a trolley. The whole system is travelling on level ground at 15 ms^{-1} . The trolley reduces speed from 15 ms^{-1} to 5 ms^{-1} in 5 seconds. The mass of the box is 20 kg. As the trolley decelerates, the frictional force between the box and the trolley is 25 N.

Describe, in detail, the motion of the box.

5. A shunting train pushes trucks along a level line. The mass of the locomotive is 20 tonnes and the mass of each truck is 4 tonnes. The locomotive can develop a force of 2000 N.

Given that the locomotive pushes one truck and ignoring any resistances,

- calculate the acceleration of the locomotive,
- calculate the thrust on the truck.

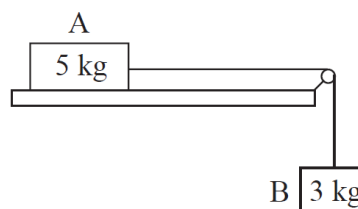
6. The diagram shows a mass A of 5 kg initially at rest on a horizontal table. A resistance force of 10 N acts against the motion of A which is connected to mass B of 3 kg by a light, inextensible string.

The system is released from rest.

- Calculate the acceleration of A.
- Calculate the tension in the string.

After a short time, B reaches the floor.

- Calculate the acceleration of A now.



7. A breakdown truck tows a car of mass 1200 kg. Calculate the tension in the tow rope if the car is

- accelerating at 0.5 ms^{-2} and experiencing a resistance force of 500 N;
- travelling at constant speed but experiencing a resistance force of 400 N.